SOLUTIONS TO IYPT PROBLEMS

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2015 Problem 17 : Coffee Cup

How to Walk with a Cup of Coffee

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Abstract

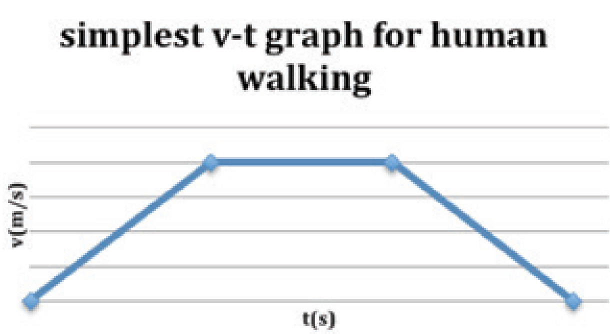
We drink at least four times a day. However, if we fill a glass with a liquid, such as coffee or water, and start to walk, it may start splashing. Our purpose is to find a theoretical model for this phenomenon. So we can analyze the problem and find a way to prevent liquid from splashing. First step is to simplify the problem. In order to do that, we will make some assumption for human walking and liquid in the glass, then, based on each of them, we will suggest three different models. After that by calculation we will find out the liquid behavior. At the end, we will design a glass that minimizes the chance of liquid splashing.

Experiment

1. Models for human walking

Following sentences describe the first suggested model. We assume the person in our problem walks between two points like a block, and the glass is attached to the block. The simplest v-t graph (velocity relative to time) for a block to move from one point to another- with initial velocity equal to zero- is shown in the chart 1. First, the block increases its velocity with a constant acceleration. Then, it may keep on moving with a constant velocity for a while. And at least in the last part, it decreases its velocity with a constant acceleration to stop at the destination.

Chart 1. v-t graph for our first model



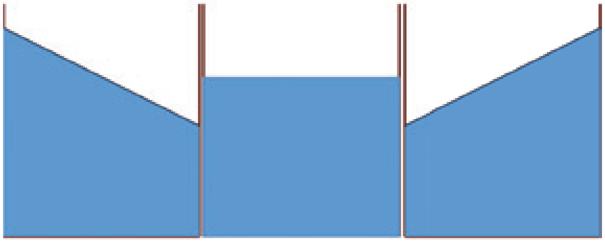


Fig.1. shows shape of surface in three different condition, a – positive acceleration, b- constant velocity, c- negative acceleration

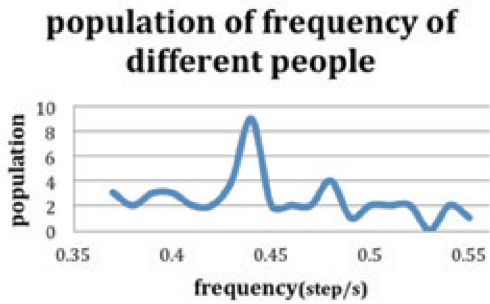
Now we study the liquid in the glass. In the first part (positive acceleration) the liquid will have a positive steep α (fig1-a). In the next part after the block reaches its maximum velocity, the liquid start oscillating about its equilibrium point (α=0). Finally in the last part, block has a negative acceleration and liquid will have a negative steep (in the theory section we will explain these three condition with more detail)

On the other hand, if we look at human walking more accurately, we find out that the body state repeats after each step. In conclusion, we can ascribe it a frequency.

Following sentences describe this model for human walking. We assume that a person walks with a constant frequency (reverse of the duration between one step to another). Beside, we insert an impulse to the glass in every step that we take; we can model the force of this impulse as Eq.2 (in which F is the force, A is a constant that can be measured through an experiment and f_h is the frequency of human walking).

In some occasions while walking, we see the liquid in the glass starts to oscillate and its amplitude is increasing over the time; this phenomenon happens when Eq.1 applies to the frequency of human (f_h) and frequency of liquid (f_l). At this point, the amplitude of liquid increases until it splashes. This is exactly what most people hate about walking with a glass filled with liquid.

Chart 2. data of human frequency (between 10-50 years old)



$$f_l = f_h \text{ or } \omega_h = \omega_l \quad (1)$$

$$F = A \sin(2\pi f_h t) \quad (2)$$

2. Experiment No.1

We tried to collect data from different people to find out an acceptable range for human walking frequency. We went to a public place. Then, we measured n times (n varied from 10 to 20) of someone's steps with a chronometer. The reason of counting more than one step was to reduce the error and find a medium frequency of a single person. After that, we repeated this experiment for as many persons as we could (more than a hundred) in the age range of 10-50 years old. Chart 2 shows the scattering of frequency over population.

3. Experiment No.2

To find f_l we needed a setup that camera and glass does not move relative to each other. So, I designed a setup that is shown in fig.3To collect data, we kicked the cart to make the liquid oscillate; and recorded a video. Later, we analyzed the video in the tracker

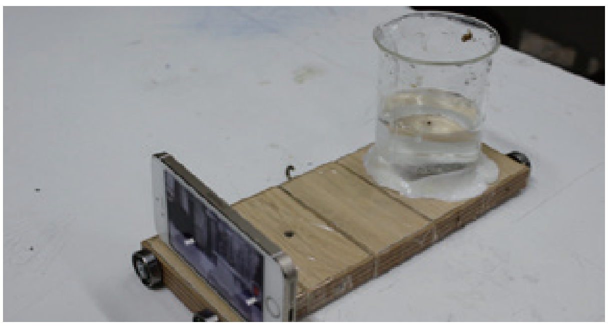


Fig.3. setup of experiment No.2

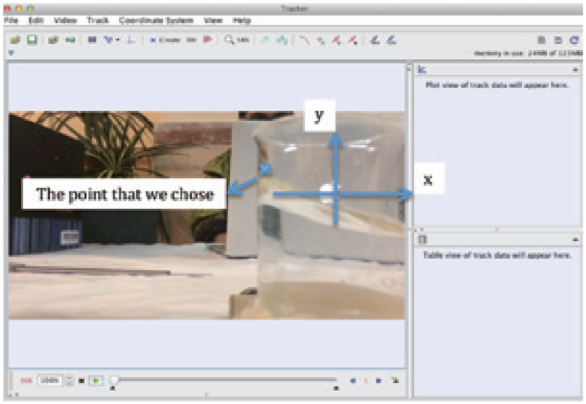


Fig.4. Tracker

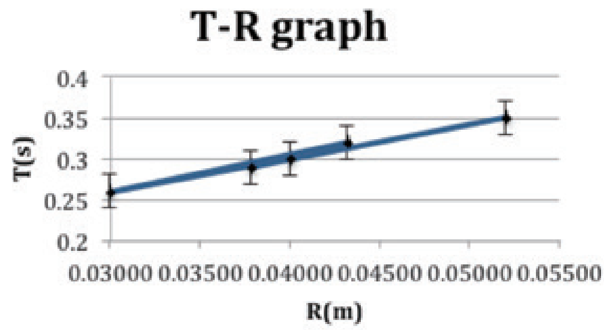


Chart 3. T (time of periodicity) in terms of R in a constant height (h=(6.00±0.02)10^{-2} m)

application and extracted information that we needed.

This program extracts data from the inserted video and then plots them in x-t and y-t chart fig.5 shows data from the video in fig.4.

In Tracker, first we insert a video and then choose the coordinate, this program asks you to choose a point mass (shown in fig.4), and after exact period of time asks you to denote that point again. Thus, we repeat this task as many times as needed. At the end, we plot these data in a chart (shown in fig.5). Then, by subtracting T1 from T2 we can easily find out the period of oscillation.

In this experiment, period of oscillation (T) in different R and equal h had been found out. We have done this experiment for radiuses in the range of glasses we usually drink with (3-6 cm). To be more accurate and reduce errors, we calculated T for each radius, 5 times and we used the averaged value in chart-3.

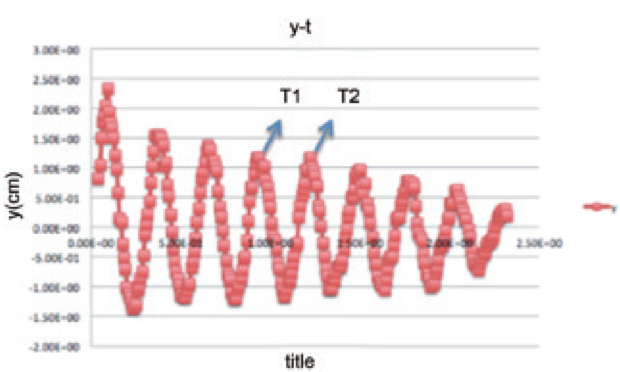


Fig.5. finding T (period of oscillating) (T = T2 - T1)

Chart 3 shows that T is a linear function of R with the r =0.996 (regression), B = (4.2 ± 0.2)($\frac{s}{m}$) (the slope). In the next part we will find the exact function of T for this range of R (glass radius).

Theory

First of all to solve a complicated problem, you should simplify it. Parameters that are effective in our problem are ρ (density of our liquid), σ (surface tension), γ (adhesion), η (viscosity), h (height of the liquid), R or d (R the radius of our glass or cup if it is cylindrical and d side of the glass if it is cubic), g (gravity of earth), a (the acceleration of the person), f_h (frequency of human walking), and M (mass of liquid).

Now from eq.3 [1] we can calculate the shape of liquid with given initial condition. In which p in the equation represents pressure.

$$\nabla p = \rho \vec{a} \qquad (3)$$

If we walk with constant velocity($a_x = 0$) so $\vec{a} = -g\hat{z}$ then by solving the differential equation (Eq.3) and assuming the atmosphere pressure equals to p_0 the result will be like Eq.4.

$$p = -\rho g z + p_0 \qquad (4)$$

As we can conclude from Eq.4 isobar surfaces have same height.

Now, if we walk with constant acceleration $a=-g\hat{z}+\hat{x}$, by resolving Eq.3 for this acceleration and same assumption as before for atmosphere pressure, the result will be like Eq.5.

$$\frac{\partial p}{\partial x}\hat{x} + \frac{\partial p}{\partial z}\hat{z} = p(-g\hat{z} + a\hat{x})$$

$$\Rightarrow p = p_0 - \rho g z + \rho a x \tag{5}$$

So the isobar surfaces equation will be like Eq.6 (here p_1 represents the pressure of this surface).

$$p_1 = p_0 - \rho g z + \rho g x$$

$$\Rightarrow x = \frac{p_1 - p_0}{\rho a} + \frac{g}{a} z \tag{6}$$

Isobar surfaces have the slope (α) where tan(α) = a/g

These are acceptable if we neglect σ and γ. For liquids that we usually drink (water, coffee, tea and etc.) σ is small enough to be neglected. Furthermore, γ works like friction and damp the oscillation so does not effect our pervious calculation.

Now we are supposed to calculate the period of a liquid oscillation in a glass. It is easy to calculate the period for a cubic glass. Eq.7 shows T_{cubic} and Eq.12 shows position of center of mass and amplitude relative to time. The solution for founding T_{cubic} explained in the index section.

$$T_{cubic} = 2\pi\sqrt{\frac{d^2}{12gh}} \tag{7}$$

In Eq.8 we assumed initial conditions are $t_i = 0$, x (t_i) = 0 and $\dot{x}(t_i) = \frac{I}{M}$, where I is the impulse that a person gives to the liquid.

$$\delta x_{cm}(t) = \frac{IT_{cubic}}{M2\pi}\sin(\frac{2\pi}{T_{cubic}}t)$$

$$\delta y(t) = \frac{6hIT_{cubic}}{dM2\pi}\sin(\frac{2\pi}{T_{cubic}}t) \tag{8}$$

From above equations we can conclude the relation between maximum amplitude and impulse is like Eq.9.

$$\delta y_{max} = \frac{6hIT_{cubic}}{dM2\pi} \tag{9}$$

To find the more accurate answer we must add the friction force in our solution as well. We can assume that friction force is a function of velocity. Then, we can write the simplified form of the force like Eq.10. Then we add this force and resolve the differential equation. The result is shown in Eq.11. As we can see the amplitude is reducing over the time.

$$\frac{F}{m} = \frac{2\pi}{T_{cubic}}(\gamma / 2\rho g)\delta\dot{y} \tag{10}$$

$$\delta x_{cm} = \frac{IT_{cubic}}{M2\pi}e^{-\frac{2\pi}{T_{cubic}}t(\gamma/\rho g)}\sin(\frac{2\pi}{T_{cubic}}t)$$

$$\delta y = \frac{6hIT_{cubic}}{dM2\pi}e^{-\frac{2\pi}{T_{cubic}}t(\gamma/\rho g)}\sin(\frac{2\pi}{T_{cubic}}t) \tag{11}$$

Until now, we have done all of our calculation for a cubic shape glass. However, most of glasses we usually drink with are cylindrical. Unfortunately, finding the period for cylindrical glass is not as simple as a cubic glass. Therefor, we are going to find the period of a liquid oscillation in a cylindrical glass by dimensional analysis.

$$T = C\sqrt{\frac{R^2}{gh}}f(\frac{\sigma}{\rho g}, \frac{R}{h}) \tag{12}$$

Eq.12 is the result of our calculation. In this equation C is a constant, and $f(\frac{\sigma}{\rho g}, \frac{R}{h})$ is a function of $\frac{\sigma}{\rho g}$ and $\frac{R}{h}$. One way to determine them is to design some experiments and review the result of them.

If we neglect σ, Eq.12 will be simplified to Eq.13

$$T = C\sqrt{\frac{R^2}{gh}}.f(\frac{R}{h}) \tag{13}$$

In this equation C is a constant that only can be determined experimentally. Furthermore, we will define $f(R/h)$ through the experiment No. 2. As we may conclude from Chart 3, T is a linear function of R. Thus, $f(R/h)$ must be a constant value; we assume it is equal to 1. Now we can find C by using the slope of the chart 3 (B).

$$C = \frac{B\sqrt{gh}}{2\pi} = 0.52 \pm 0.02 \tag{14}$$

Also, we can rewrite Eq. 13 and use the value of C in the Eq.14. Eq. 15 shows the final result.

$$T = (0.52)2\pi\sqrt{\frac{R^2}{gh}} \tag{15}$$

If the frequency of walking and liquid were the same, the equation for the amplitude of liquid differs from the Eq.11. After resolving the equation, Eq.16 shows the amplitude of liquid relative to time.

$$\delta y = \frac{6hI}{dM\omega}\, e^{-\omega_h t(\gamma/\rho g)} \sin(\omega_h t) + \frac{A}{2}\, t\sin(\omega_h t) \tag{16}$$

Now we can conclude that if the frequency of walking and liquid were the same, the amplitude of liquid oscillation increases over the time. Thus, this could be the reason of liquid splashing.

Theory Vs. Experiment

In this section our aim is to compare the theory and the experiment part. Lets start with the first model for human walking. As we can see in Eq.6 relation between slope of the liquid surface and acceleration is like Eq.17.

$$\tan(\alpha) = {}^{a}/_{g} \tag{17}$$

Hence, only a huge acceleration can cause a splash. Now the question is, what is the maximum acceleration that a person can have? We tried to find out an approximate value for the maximum possible acceleration that a human can reach. We asked people to start to run with their maximum power. Our experiment resulted with the maximum acceleration equals to 0.04 ${}^{m}/_{s}{}^{2}$.

$$\tan\alpha = \frac{2}{50} = 0.04 \rightarrow \alpha \approx 0.04\ rad \tag{18}$$

Eq.18 shows that the maximum slope that the surface of a liquid can reach is equal to 0.04 *rad*. If we assume the radius of the glass is equal to 3cm; this means that the maximum amplitude of the liquid is approximately equal to 1(mm). This is smaller than something liquid may reach while splashing. So this model is not efficient.

Next model concludes that if the frequency of walking and liquid were the same it would cause resonance. Thus, we found a range for human walking frequency in chart 2. f^{-1} = (0.37-0.55)(s/step). Then, we found relation between frequency of liquid, height of the liquid and radius of the glass in Eq.15. Moreover, for normal glasses we usually drink with we can assume R = (3.0±0.1) cm so h must be less than (8.4 ± 0.1) cm and more than (15.2 ± 0.1) cm to avoid resonance. Furthermore, we recommend glass factories to use this calculation and find the best height for glasses they produce.

At the end, if none of above claims was applicable and yet we observe a splash; then from Eq. 9 we must have inserted an enormous impulse in a step.

Reduce splashing

Now we know why liquid splashes while walking. Our aim in this part is to find a way to reduce or ignore splash. In order to do that, we must cancel or decrease the source of it. One way is to change the height of liquid to decrease the chance of splashing and another way is to build our glasses like fig-6.

Now we can determine relation between constants introduced in the fig-6. First, we should find the relation between $\boldsymbol{\delta}$, $\boldsymbol{\delta_1}$, $\boldsymbol{\alpha}$, $\boldsymbol{a_{max}}$, g, R (*radius of the glass*). In the following equations $\boldsymbol{y_1}$ is the highest point of the liquid surface in a specific x, and $\boldsymbol{y_2}(x)$ is the function of the glass surface relative to x.

$$y_1(x) = -\delta + \frac{\delta y_{max}}{R}x \tag{19}$$

$$y_2(x) = \alpha x^2 \tag{20}$$

$$y_2(R) - y_1(R) = \delta_1 \tag{21}$$

From Eq. 19-21

$$\alpha R^2 + \delta - \frac{\delta y_{max}}{R} R = \delta_1$$

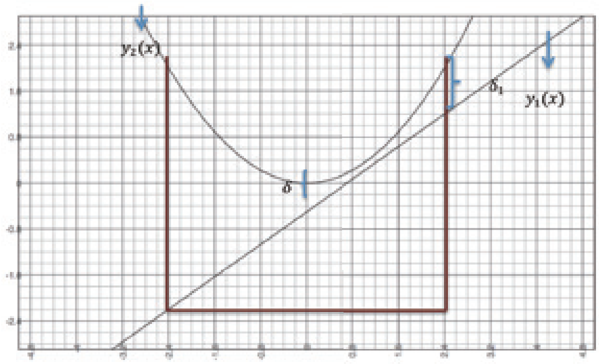


Fig.6. shape of the glass in 2D where $y_1(x) = -\delta + {}^{a_{max}}/_{g}\,x$ and $y_2(x) = \alpha x2$

δ Is the distance between liquid and lowest point of the glass (x = 0, y = 0), $\boldsymbol{\delta y_{max}}$ is the maximum amplitude that we found from the data. At the end, by putting the numerical value of each parameter we can find α and the shape of the glass.

Moreover, there are two things that ignore splashing, stirring liquid or using a baffle in the glass. Something that we must notice is that nobody stirs a liquid such as water! Thus it is safer and even more general to use baffles in our glasses. Fig.7 shows an example of a baffle in a glass.

Sometimes we may see glasses like the one shown in the fig-8. These glasses damp liquid oscillations because, their edges work like a damper or a baffle. Therefore, they are considered as a better choice for those who usually walk with a glass in their hand.

Conclusion

We tried to discover effective facts on splashing a liquid in a glass that you are carrying while walking. Furthermore, we suggested few theoretical models for human walking and explained what will happen for liquid in each of them. Then, we determined what causes liquid to splash. Later, we took advantage of our discoveries to design glasses that reduce or ignore this phenomenon. Thus, we designed glasses that liquid does not splash in them or splashes less than normal glasses.

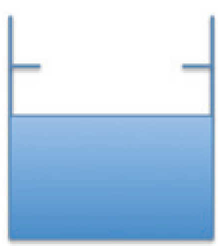


Fig.7. using a baffle to ignore splashing

Fig.8. using a baffle to ignore splashing

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Now we are going to explain how we found out T in Eq.5

Eq.22 $\overrightarrow{r_{cm}} = \dfrac{\int dm\,\vec{r}}{M}$

From Eq.22 we can find out x_{cm} [2]

Eq.23 $x_{cm} = {}^{\iint \sigma\, dx\, dy\, x}/_{M} = {}^{d}/_{6h}\,\delta y$

So the changes in x of center of mass (δx_{cm}) of this cubic glass is

Eq.24 $\delta x_{cm} = \dfrac{d}{6h}\,\delta y$

By changing the center of mass the pressure on the right and left side changes so the water will oscillate, we will find out Σf (from Eq.25)

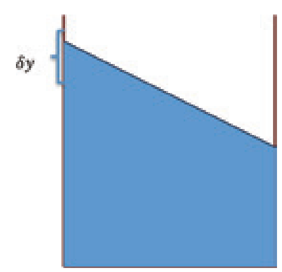

Fig.9. liquid preverse from equillbrium position about δy

$$\Sigma f = f_r + f_l = -\int_0^{h+\delta y} \rho g dy\, dy - \int_0^{h-\delta y} \rho g dy\, dy$$

$$= -2\rho g h\, \delta y \qquad (25)$$

From Eq.25 and Newton seconds law ($\Sigma f_{ext} = m a_{cm}$ where Σf_{ext}=Σf in Eq.25 and $a_{cm} = \delta \ddot{x}_{cm} \ddot{x}$)

$$\Sigma f = -2\rho g h\, \delta y = \rho h d^2 \left(\frac{d}{6h}\, \delta \ddot{y}\right) \qquad (26)$$

General solution for above differential equation (Eq.26) is Eq.27 [3]

$$\delta y = A \sin(\omega t + \varphi) \qquad (27)$$

Where $\omega = \sqrt{12gh/d^2}$, A and φ are arbitrary constants that can be chosen to make the general solution meet any two given independent initial conditions.

References

[1]-Fluid mechanics, Amir Aghamohamadi

[2]-An introduction to mechanics- Daniel Kelepner

[3]-Calculus and Analytic Geometry- George B. Thomas, Ross L. Finny