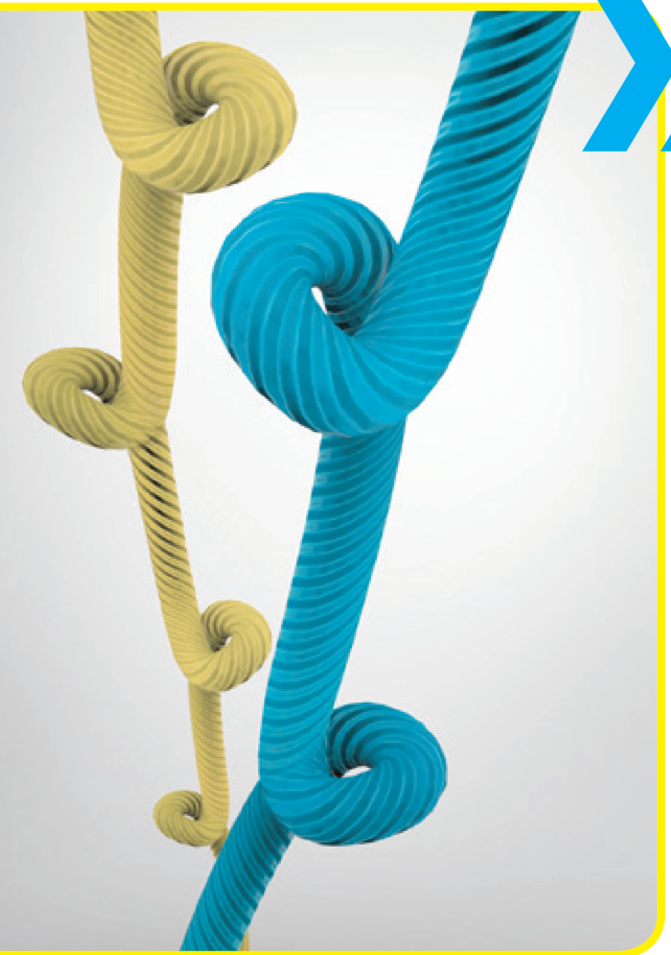

## 2014 Problem 3 : Twisted Rope

### When twisted, the rope will form a helix or a loop


### Abstract

Twisting a piece of string at some point causes the string to become buckled. Further twisting that piece of string will eventually cause it to coil around itself forming a Helix-Like structure. In this research, we investigated the formation of the first loop, and measured the distance of the two ends of the rope towards each other ( D ) and the rotation angle ( R ). Displacements are controlled and the corresponding forces and moments remain passive. Several plastic ropes with different lengths are used. One of the ends of the ropes was fixed and the other end was rotated. Once the first loop occurred, the distance between the two ends of the ropes was measured. The experiment's error was nearly 10% and we only surveyed the mathematical model in two dimensions. The gravity effect is neglected. We concluded that, by using the mathematical model, we can understand when a loop occurs with regards to specific (R)s and (D)s.



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### Introduction and theory

Twisting a piece of string at some point causes the string to become buckled. Further twisting that piece of string will eventually cause it to coil around itself forming a Helix-Like structure. This is something that all of us have observed at some point in our real life. Born , carried out some elegant large deflection bending experiments by hanging weights on the end of a rod (i.e., dead loading)[1]. Yabuta,using an energy method, assumes an initial helical deformation (which is Love's solution[3]) and obtained the Greenhill,formula for the onset of looping, which in fact describes the primary bifurcation for a rod with zero bending moments at its ends. Modeling the loop as a circle, he also derived a formula for the point at which it reopens (i.e., pop-out), which he compared with his experimental results[2]. Goss and coworkers by using varied R ( Rotation ) and fixed D (slack ), found that if a loop forms in a rod, then unwinding the twist may instigate a dynamic jump as the rod pops out of self-contact[4]. The importance of this phenomenon is comprehended in several aspects of science. For example: In **"engineering"** , Marine cables under low tension and torsion on the sea floor undergo a buckling process during which tensional energy is converted to flexural energy . The cable becomes highly contorted with loops and tangles, this can permanently damage the cable. In **"textile industry"**, and the study of multi-filament structures such as yarns is interesting.

In **"biology"** rod models are used to describe the supercoiling of DNA molecules .**DNA supercoiling** refers to the over- or under-winding of a DNA strand, and is an expression of the strain on that strand. Supercoiling is important in a number of biological processes, such as compacting DNA. Additionally, certain enzymes such as topoisomerases are able to change DNA topology to facilitate functions such as DNA replication or transcription.

It is a common observation that when a rope is twisted it tends to form loops or coils. This effect has been cited in physics to explain the instability of twisted magnetic fields [5]. If a DNA segment under twist strain is closed into a circle by joining its two ends and then allowed to move freely, the circular DNA would contort into a new shape. Such a contortion is a **supercoil** (Fig. 1).

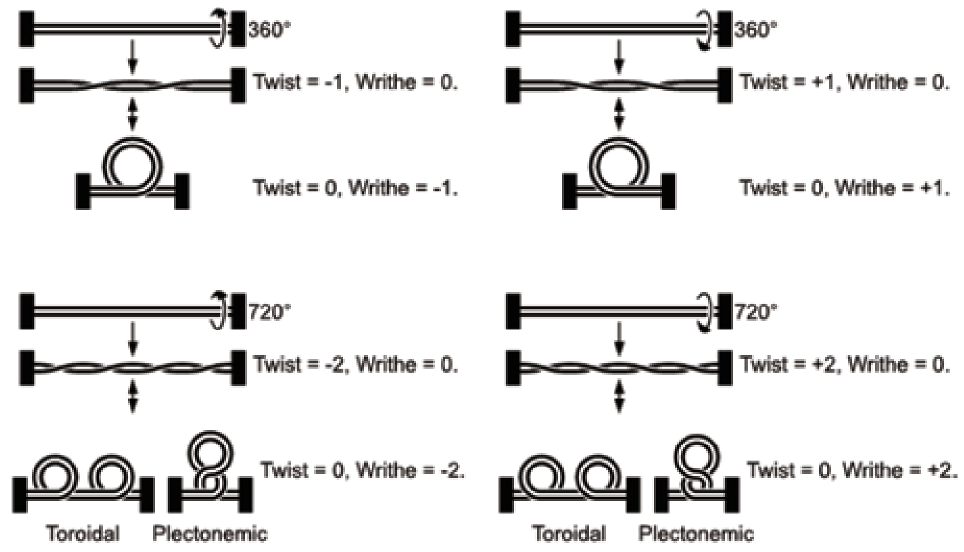


*Figure 2 -Supercoiled structure of linear DNA molecules with constrained ends.*

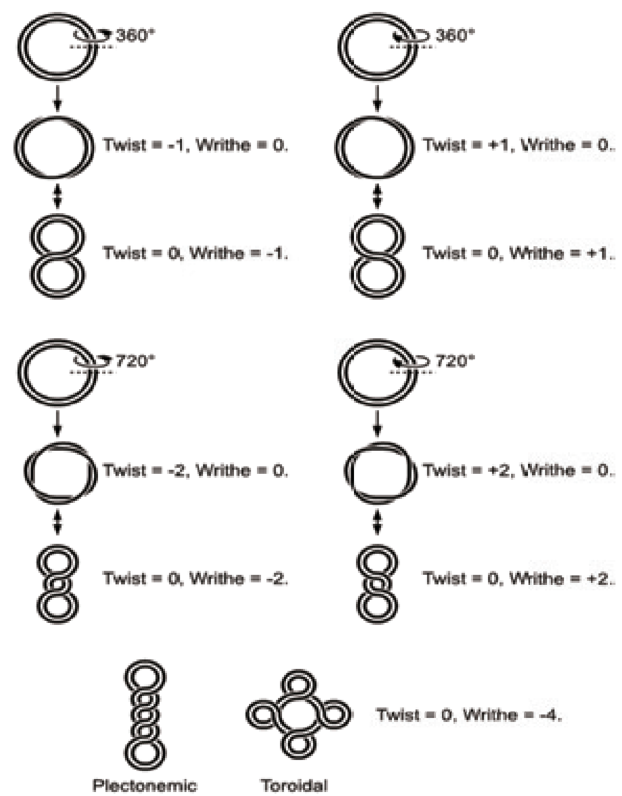


*Figure 1-Supercoiled structure of circular DNA molecules with low writhe. Note that the helical nature ofthe DNA duplex is omitted for clarity (Wikipedia).*

**A superhelix** is a molecular structure in which a helix is itself coiled into a helix. This is significant to both proteins and genetic material, such as overwound circular DNA. Contrary to intuition a topological property, the linking number, arises from the geometric properties twist and writhe according to the equation (1):

$$L_k = T + W \qquad (1)$$

Where $L_k$ is the linking number, **W** is the writhe and T is the twist of the coil. In mathematics, the **linking number** is a numerical invariant that describes the linking of two closed curves in three-dimensional space. In DNA this property does not change and can only be modified by specialized enzymes. Intuitively, the linking number represents the number of times that each curve winds around the other. The linking number is always an integer, but may be positive or negative depending on the orientation of the two curves. The linking number was introduced by Gauss in the form of the **linking integral**. Any two closed curves in space, if allowed to pass through themselves but not each other, can be moved into exactly one of the following standard positions. This determines the linking number (Fig. 3).

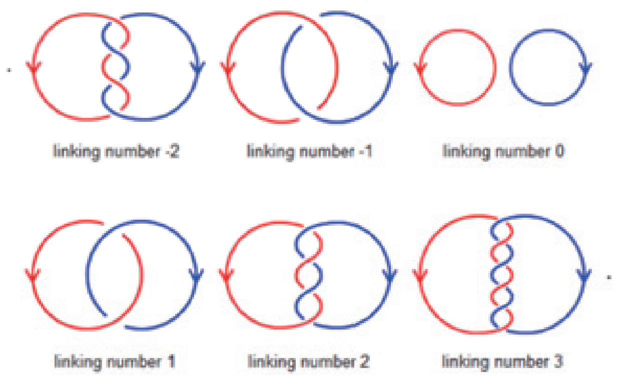


*Figure 3- Here on the left, Linking Number is found by counting thenumber of crossings and dividing by two(Wikipedia).*

There is an algorithm to compute the linking number of two curves. Label each crossing as positive or negative, according to the following rule (Fig. 4).

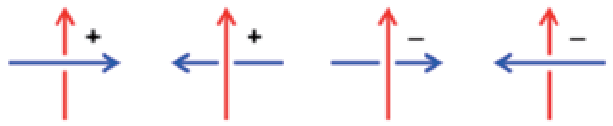


*Figure 4-This Figure shows us, each crossing is positive or negative (Wikipedia).*

The total number of positive crossings minus the total number of negative crossings is equal to twice the linking number (Eq. 2).

$$L_k = \frac{n1\ +\ n2\ -\ n3\ -\ n4}{2} \qquad (2)$$

Where $n_1$, $n_2$, $n_3$, $n_4$ represent the number of crossings of each of the four types.

The two sums **n1 + n3** and **n2 + n4** are always equal, which leads to (Eq. 3).

$$L_k = n_1 - n_4 = n_2 - n \qquad (3)$$

Note that **n1-n4** involves only the under crossings of the blue curve by the red, while **n2-n3** involves only the overcrossings. Experience teaches us that the response of a rod depends on the material it is made of, the geometry of its cross-section, the manner in which it is held at its ends, the type of loading, and the loading sequence.

In this research, we investigated the formation of the first loop , and measured the distance of the two ends of the rope towards each other ( D ) and the rotation angle ( R ). the displacements are controlled and the corresponding forces and moments remain passive.

All real "rigid" bodies are to some extent elastic, which means that we can change their dimensions slightly by pulling, pushing, twisting or compressing them (Fig. 5).

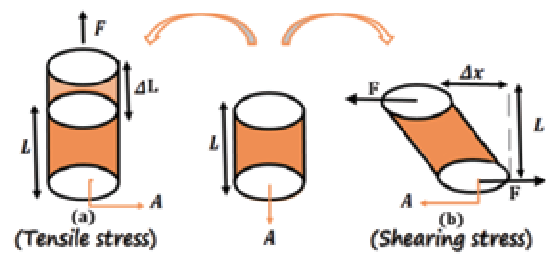


*Figure.5-(a) A cylinder subject to tensile stress stretches by an amount of* ***ΔL*** *(b) A cylinder subject to shearing stress deforms by amount* ***ΔX****.*

Figure 5 shows two ways in which a solid might change its dimension when forces act on it. In state (a) A cylinder is stretched in state(b) A cylinder is deformed by a force perpendicular to its long axis.

What the two deformation types have in common is that a **stress**, deforming force per unit area, produces a strain, or unit deformation. In Figure.5A tensile stress is illustrated in state (a) A shearing stress in state(b). The stress and strain take different forms in these two situations but they are proportional to each other and the constant proportionality is called a **Modulus of elasticity**. So (Eq. 4):

$$\textbf{Stress = Modulus * Strain} \qquad (4)$$

For a range of applied stresses, the stress-strain relation is linear (the subject reverses to it first situation when the stress is removed). If the stress is increased beyond the **Yield strength** of the object, the subject becomes permanently deformed and if the stress continues to increase, the object eventually ruptures at a level of stress called, the **Ultimate strength**. That's the reason that an elastic rope is used, because in elastic objects, the level of yield strength is higher than others. An elastic object is an object that, every section of it experiences the same strain when a given stress is applied but in other objects the strain that each particle experience isn't equal and it's erratic. Also the level of elasticity wasn't so high in ropes, so massive stresses couldn't be applied to the ropes because they deformed and caused bigger errors in our experiments.

## Tension

For tension and compression, the stress on the objects is defined as F/A, where F is the magnitude of the force applied **perpendicular** to an area A on the object. Strain is a dimensionless quantity ***ΔL/L*** (the fractional or percentage changes in length of the object). Because the strain is a dimensionless the modulus has the same dimension as the stress – namely force per unit area. The modulus for tensile and compression stresses is called the **Young's modulus** and is represented by the symbol **E** or **Y** (Eq. 5).

$$\frac{F}{A} = E\frac{\Delta L}{L} \qquad (5)$$

## Shearing

In shearing, the stress is a force per unit area, but the vector lies in the plane of the area. The strain is a dimensionless ratio ***ΔX/L***. The corresponding modulus which is given the symbol ***G*** or ***μ*** is called **Shear Modulus** (Eq. 6).

$$\frac{F}{A} = G\frac{\Delta x}{L} \qquad (6)$$

Elastic objects reverse to their first situation if their stress is removed (if stress is under the yield strength), the stress-strain relation is linear so they are under the influence of a Hook's law restoring force given by **F=-kx**. So we can measure the Energy of shearing and tensile force by combining Hook's law with( Eq.5) and (Eq.6) :

$$Energy = \frac{1}{2}kx^2 \qquad (7)$$

$$Tensile\ Energy = \frac{1}{2}\frac{AE}{L}\Delta L^2 \qquad (8)$$

$$Shearing\ Energy = \frac{1}{2}\frac{AE}{L}\Delta x^2 \qquad (9)$$

We can calculate the energy and the torque needed to twist a rope of length L through an angle $\varphi$ by considering the radius of the rope (r).

In Figure ( 6) we calculate the exerted torque of a part of a cylinder. And then we can add up the torques of all the particles to find the torque of the body as a whole. The sum is taken over all the particles in the body (Eq. 10, 11).

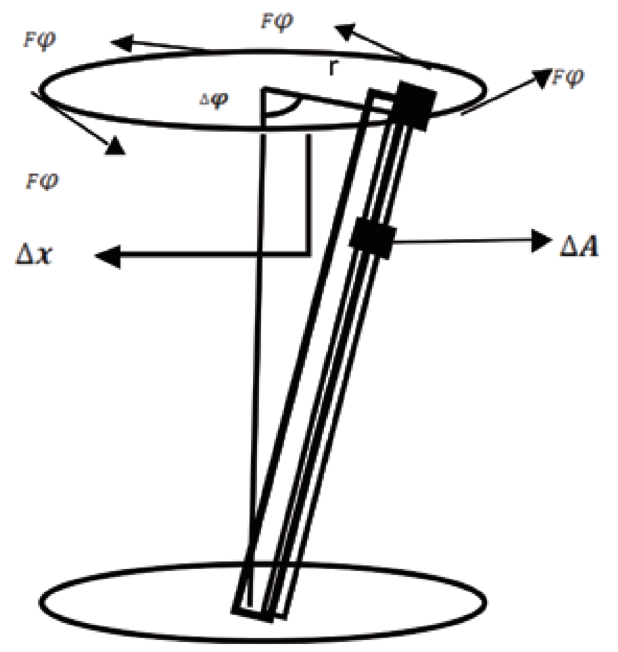


*Figure 6. We measure the angular position* $\Delta\varphi$ *by this equation* $\Delta\varphi=\frac{\Delta x}{r}$ *where* $\Delta x$ *is the arc length of a circular path of radius r and angel* $\Delta\varphi$ *.and* $F\varphi$ *is the force of a part of the cylinder.*

$$\frac{F\varphi}{\Delta A}=\mu\frac{\Delta x}{L}=\mu\frac{AE}{L} \qquad (10)$$

$$F\varphi=\Delta A\,\mu\frac{r\Delta\varphi}{L} \qquad (11)$$

Where $\mu$ is the shearing modulus.

The body rotates about a rotation axis, changing its angular position from $\varphi_1$ to $\varphi_2$ under goes an angular displacement $\Delta\varphi$.

Now we want to calculate the amount of torque exerted on a part of the elastic cylinder for an angular displacement $\Delta\varphi$ around the rotation axis.

$$\Delta m=rF \qquad (12)$$

Combining ( Eq. 11) and (Eq.12) gives us (Eq. 13):

$$\Delta m=rF\varphi=\mu(\Delta A)\frac{r\Delta\varphi}{L} \qquad (13)$$

Where $\Delta m$ is an amount of torque for an angular displacement $\Delta\varphi$ in a part of the cylinderandF is the applied force.

$$M=\sum\frac{\mu r\Delta\varphi}{L}(\Delta A) \qquad (14)$$

$$\Delta A=\Delta r(\Delta\theta r) \qquad (15)$$

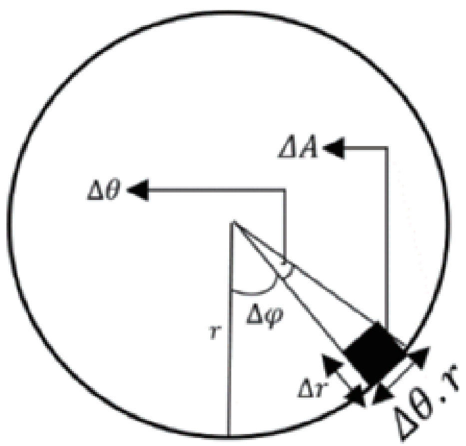


*Figure 7. Figure 6 view from top*

The second moment of area that, designated by 'I', tells us how the cross-section of the rotating body is disturbed about an axis. 'I', depends on how the load is applied to the cylinder. When we are twisting a cylinder 'I' is:

$$I=\sum r^3 2\pi\Delta r\frac{\pi r^4}{2} \qquad (16)$$

So combining the Eq.15 and Eq.16 is:

$$M=\frac{\mu\Delta\varphi}{L}\frac{\pi r^4}{2}=\frac{\mu\Delta\varphi}{L}\mathbf{I} \qquad (17)$$

and the energy is:

$$E=\frac{1}{2}M\Delta\varphi \qquad (18)$$

Using Eq.17 and Eq.18 we can rewrite the energy:

$$E=\frac{1}{2}M\Delta\varphi=\frac{\mu\Delta\varphi^2}{L}\mathbf{I} \qquad (19)$$

We found that a simple beam is subject to a shearing force and a bending moment along its length, both of which tend to distort it from its straight unloaded shape. But experience shows that the distortion due to shear in a beam which its length is much greater than its thickness is completely negligible compared with the distortion produced by the action of the bending moment.

In real world if we wanted to measure the amount of B or either Young modulus or Shearing modulus of the rope , first we attach a short rope in to a wall like Figure8, after the rope tilted by the gravity .

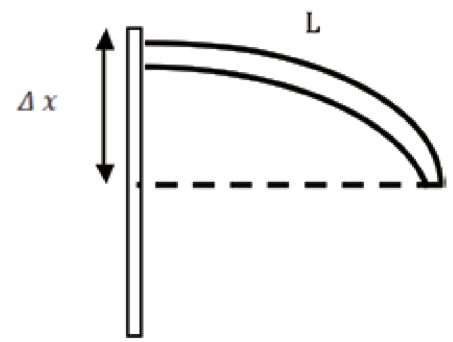


*Figure 8– A rope that is attached to wall is tilted by gravity .Where* $\Delta x$ *is the distance of the end of the rope before and after tilting.*

We measure the $\Delta x$ and afterwards we use this formula:

$$\Delta x=\frac{WL^4}{8B} \qquad (20)$$

where B is Bulk modulus that is B=E.I and W is the weight per unit of length.

The amount of G (shearing modulus) is calculated using (Eq. 21).

$$Y=2\mu(1+\sigma) \qquad (21)$$

Where Y is the Young modulus, $\mu$ is the Shearing modulus and $\sigma$ Poisson's ratio.

Poisson's ratio is the negative ratio of transverse to axial strain. When a material is compressed in one direction, it usually tends to expand in the other two directions perpendicular to the direction of compression. This phenomenon is called the **Poisson effect**.

Poisson's ratio $\sigma$ is a measure of this effect. The Poisson ratio is the fraction (or percent) of expansion divided by the fraction (or percent) of compression, for small values of these changes.

Conversely, if the material is stretched rather than compressed, it usually tends to contract in the directions transverse to the direction of stretching. This is common observation when a rubber band is stretched, when it becomes noticeably thinner.

## Materials and Methods

In our experiments we used several plastic ropes with different lengths. We fixed one end of the ropes and rotated the other end. Once the first loop occurred, we measured the distance between the two ends of the ropes. Afterwards we made charts with these measurements.

*Figure 9- setup view*

We wanted the energy to be a dimensionless quantity so we factorize the phrase "$\frac{1}{2}\frac{C}{L}$" and the Energy of state (b)is:

$$E_b=\frac{1}{2}C(\frac{\varphi}{L})2+\frac{1}{2}\frac{B}{R^2}L \qquad (22)$$

$$E_b=\frac{1}{2}\frac{C}{L}\left(\frac{B}{C}\frac{1}{(\frac{R^2}{L^2})}+\varphi\right) \qquad (23)$$

Then we drew the charts by using "Matlab" program.

After that we found out that with a specific d (distance of the two ends of the rope) as it explained in ( Fig. 10). The magnitude of Energy (a) and (b) changes and in one point the two energies become equal.

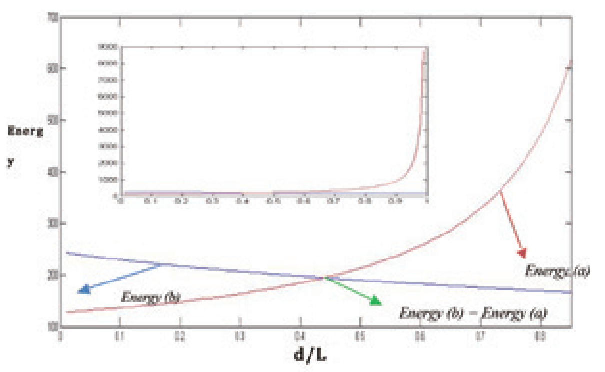


*Figure 10.-when the distance of the two ends of the rope is equal to the length of the rope (d=L) after we reduce d, the Energy of state(a) starts to increase and the Energy of state (b) starts to decrease , until the Energies are at the same level in magnitude , but after that the Energy of state(b) starts to be bigger than Energy of state(a) at this time the rope suddenly changes into state(a) from state(b) and that's the time that we have a loop in our rope .*

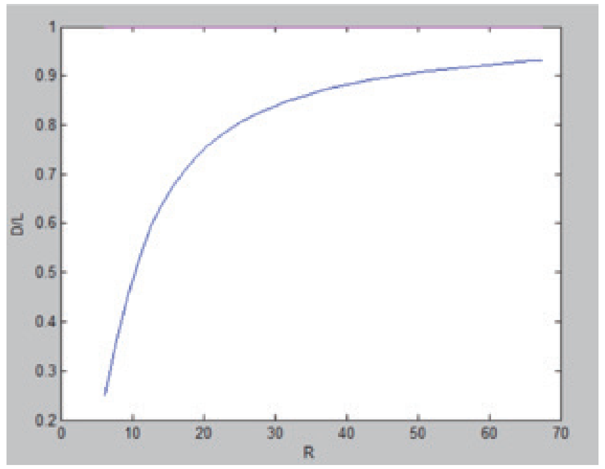


*Figure 11. In (Fig. 1) we found out how much energy is saved in the rope when a loop occurs. So we discovered the amount of d and R in each point*

And at last we compared our experiments and the mathematical model with each other.

## Conclusion

After we compared our experiments with mathematical model we found out that these energies are approximately equal to each other (Fig. 12).

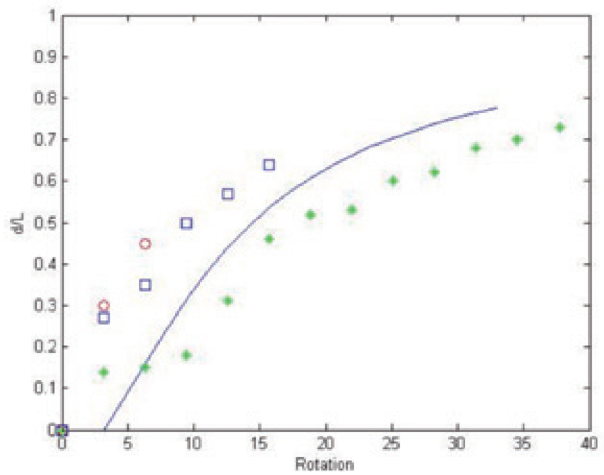


*Figure 12. Comparison of our experiments and mathematical model*

As you can see in Figure (12) the curved line is our mathematical chart and the others are, our experiments. The Experiment's Error was nearly 10% and we only surveyed the mathematical model in two dimensions.

We concluded that, by using the mathematical model, we can understand when a loop occurs with regards to specific (R) s and (D)s.

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