2015 Problem 17 : Coffee Cup

Why does the Coffee Spill While Walking?

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Key Words

resonance; nonlinear oscillation; spherical pendulum model; liquid sloshing dynamics

Abstract

People like drinking coffee, however walking with a cup of coffee can be problematic. In this paper, we investigate how the shape of the cup, frequency of walking in both horizontal and vertical directions and other parameters affect the likelihood of coffee being spilt while walking. Furthermore, we provide out a practical method to prevent the coffee from spilling out.

Introduction

When people walk holding a cup filled with coffee, the coffee always spills out. It's a life problem of liquid sloshing dynamics but it isn't merely a life problem. The oscillation of liquid in a container was first come up with by NASA in 1967.

This problem has been studied by H.C.Mayer and R.Krechetinkov very deeply. They investigated how the motion of the people, including walking speed, the degree of concentration and the angle of the liquid, influences the probability of the coffee's spilling. Due to this work, they won the Ig Nobel Prize of Physics in 2012.

This paper is based on the research of H.C.Mayer and R.Krechetikov[1], we studied the influence of the shape of the cup, frequency of people's walking, and the existence of the tea-strainer on the probability of the coffee's spilling. Furthermore, this paper focuses more on the rotation of the coffee.

Our studies

A.Theoretical analysis

The most important reason of coffee's spilt is resonance. There will be an oscillation in horizontal and vertical directions while people are walking. When the frequency of the human's walking is close to the coffee's natural frequency, which means the least eigenvalue of the liquid dynamic equation, the coffee will terribly slosh and be likely to spill.

A forced oscillation model is built to calculate the natural frequency of the coffee.

Let us consider a rectangle container with length L, and liquid elevation H. Then two sets of coordinates are: $O_0X_0Z_0$ is a fixed coordinate and OXZ is a moving coordinate attached to the container. At the beginning, these two coordinates coincide with each other. The plane Z_0 coincides with the stable water surface. We can see the schematic here. The container is considered being in a two dimensional motion and displacement is defined as $X_b = [X_b(t), Z_b(t)]$.

For ideal fluid, the velocity potential satisfies the Laplace's equation:

$$\nabla^2\phi = 0 \tag{1}$$

At the sidewall of the container, ϕ satisfies the following equation:

$$\frac{\partial\phi}{\partial n} = U\cdot n \tag{2}$$

$U = \frac{dX_b}{dt}$ is the velocity of the container, and n is the outer normal vector of the sidewall.

At the free surface, we can get kinematic and dynamic equations in $z_0 = \eta_0(x_0, t)$

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}\nabla\phi g\nabla\phi + g\eta_0 = 0 \tag{3}$$

$$\frac{\partial\eta_0}{\partial t} + \frac{\partial\phi}{\partial x_0}\frac{\partial\eta_0}{\partial x_0} - \frac{\partial\phi}{\partial z_0} = 0 \tag{4}$$

η_0 stands for the free surface and g stands for the gravity acceleration.

According to the equations above we can get the natural frequency of the liquid.

$$\omega = \sqrt{g\pi/L\tanh(\pi H/L)} \tag{5}$$

Now, let's consider a cylindrical mug. We suppose the wall of the mug is nonflexible, and the coffee is ideal, vorticity-free and incompressible, surface of the liquid is consecutive all the time. According to the Laplace's equation and the boundary conditions, we can get these equations:

$$\nabla^2\phi = 0 \quad V \tag{6}$$

$$\frac{\partial\phi}{\partial n} = 0 \quad \partial V_\omega \tag{7}$$

$$\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial n} = 0 \quad \partial V_f \tag{8}$$

General solution $\varphi(x,y,z,t) = i\sigma\phi(x,y,z)e^{i\omega t}$ is plugged into the equations forward and we can get the eigenfunction of free sloshing :

$$\nabla^2\phi = 0 \tag{9}$$

$$\frac{\partial\phi}{\partial n} = 0 \tag{10}$$

$$\frac{\partial^2\phi}{\partial t^2} = \frac{\sigma^2}{g}\phi \tag{11}$$

The following functional can be built up :

$$L(\phi) = \frac{1}{2}\left[\int_V (\nabla\phi)^2 dv - \frac{\sigma^2}{g}\int_{\partial V_f}\phi dS\right] \tag{12}$$

Now the problem is transformed into an extreme-value problem of the functional $\delta L(\phi) = 0$

When the boundary condition of the cylinder is taken into consideration, the natural frequency of liquid in a cylindrical cup can be worked out:

$$\omega = \sqrt{\frac{g\upsilon}{R}\tanh(\upsilon\frac{H}{R})} \tag{13}$$

υ is the first zero of the derivative of the Bessel function of the first kind.

In our daily life we find there is a rotation during oscillation in the coffee cup during people's waking. We can see that the rotation will reduce the amplitude. Furthermore, the inertia can avoid the splash of liquid to some extent. We think that rotation can prevent the liquid from spilling out.

The rotation of oscillating liquid is a nonlinear motion. Because our theory couldn't explain the rotation of the liquid, so we established another model——spherical pendulum model[5].

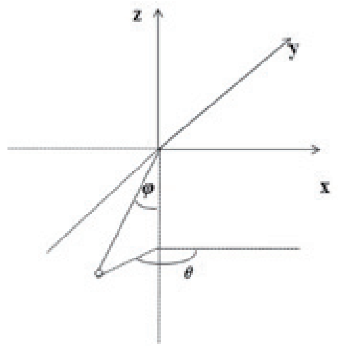


Fig.1. general coordinate

The model is a spherical pendulum with length l and mass m:

$$\begin{aligned} x &= \sin\varphi\sin\theta \\ y &= -\sin\varphi\cos\theta \\ z &= -\cos\varphi \end{aligned} \tag{14}$$

$$\boldsymbol{T}=\frac{1}{2}mgl[(\dot{x}+\dot{X}(t))^2+\dot{y}^2+\dot{z}^2\,] \tag{15}$$

$$V = mglz \tag{16}$$

$$L = T - V \tag{17}$$

Choose the φ and θ as the general coordinates. ω_{cup} represents walking frequency, and represents walking amplitude. n^* stands for the number of harmonics and the excitation is expressed by

$$X(t)=A\sin\omega_{cup}t+\sum_{n=1}^{n^*}a_n\sin(\omega_n t+\phi_n) \tag{18}$$

Solve the Euler-Lagrange equations :

$$\begin{cases} \dfrac{\mathrm{d}\left(\dfrac{\partial L}{\partial\dot{\phi}}\right)}{\mathrm{d}t}\quad\dfrac{\partial L}{\partial\phi}=0 \\ \dfrac{\mathrm{d}\left(\dfrac{\partial L}{\partial\dot{\theta}}\right)}{\mathrm{d}t}-\dfrac{\partial L}{\partial\theta}=0 \end{cases} \tag{19}$$

Natural frequency of cylindrical cup can be given out:

$$\omega=\sqrt{\frac{g\upsilon}{R}\tanh(\upsilon\frac{H}{R})}$$

Natural frequency of square cup can be given out:

$$\omega=\sqrt{g\pi/L\tanh(\pi H/L)}$$

Spherical pendulum model gives us the same result of the natural frequency but the pendulum is more proper because it is a nonlinear model and it can explain the rotation of the liquid. But the rotation of the liquid is too complex so we just give out a qualitative explanation according to documents.

When the liquid oscillates with large amplitude and the load frequency is close to natural frequency, there will be an unstable rotation regardless of whether the container has rotational symmetry. This rotation won't disappear until the excitation frequency becomes larger than its natural frequency.

In order to confirm the theory in the document, we selected 3 beakers in different sizes and a glass with square cross section.

In the chart above, ω stands for the circular frequency of the container, f means the natural frequency of the container. f(theoretical) represents the theoretical calculation of natural frequency and f(experimental) represents the results of measuring the cup's natural frequency. We can see that experimental natural frequencies correspond to theoretical calculation, so we confirmed the theory.

Table 1. parameters of containers and results of the experiments

	Diameter /side(cm)	depth (cm)	Height of water (cm)	ω	f （theoretical）(Hz)	f （experimental）(Hz)
Small beaker	4.978	6.804	4.536	26.8905	4.279	4.2
Medium beaker	6.594	9.530	6.340	23.3730	3.719	3.7
Big beaker	10.296	15.354	10.236	18.7083	2.977	2.9
Square cross section glass	5.590	13.186	8.800	23.4671	3.734	3.7

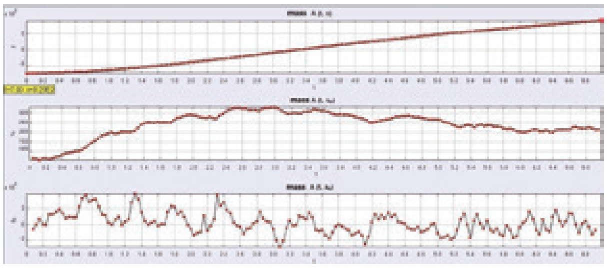


Fig.2. Walking motion in x direction

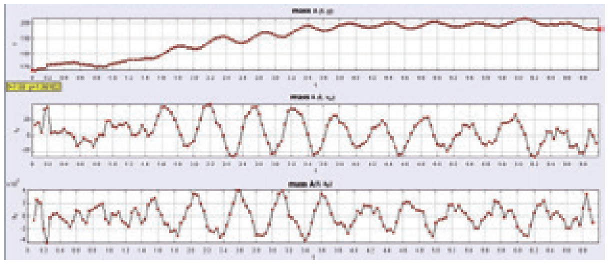


Fig.3. Walking motion in y direction

B. Studies on the motion of people

In order to research the motion of people holding a coffee cup, we shot videos by DSLR (Digital Single Lens Reflex) and analyze them by Tracker. From the Figure 2 and Figure 3, a conclusion can be drawed that the movement in x direction comes to a state with uniform speed after a uniform acceleration and in y direction there's a regular oscillation and it has a stable period. It can be disassembled into a simple harmonic oscillation and a noise.

We did a FFT on the acceleration in both x direction and y direction and the results are showed in Figure 4 and Figure 5. There are 3 peaks at 0.625,1.875 and 3.75Hz in x direction and the frequency of oscillation in y direction is 1.9Hz or so.

Then we used the oscillator to force the beaker filled with water to oscillate at 0.625, 1.875 and 3.75 Hz and the phenomenon at 3.75 Hz is sloshing with a large amplitude and a rotation. The resonance frequency we measured before is about 3.71 Hz.

So we can make a conclusion that the horizontal oscillation of the walking people does cause the resonance and makes coffee spill more likely.

Then we used the oscillator to make the beaker vibrate in y direction at 1.9 Hz. Picture on the left hand shows the phenomenon when the frequency of the driving force is 1.9Hz and there isn't enough driving force to make it slosh. Then we adjusted the output frequency of the signal generator and let the liquid reach to resonance. There wasn't a violent surge and there was no rotation in this motion.

Thus, we can draw a conclusion that the vibration in y direction during walking contributes quite little to the spilt of coffee.

C. Investigate the impact of the cups' shapes

We selected 4 cups in different shapes, and filled them with water. The water elevation equaled the two thirds of the depth. We put them on an oscillator to make them oscillate horizontally and vertically. Then

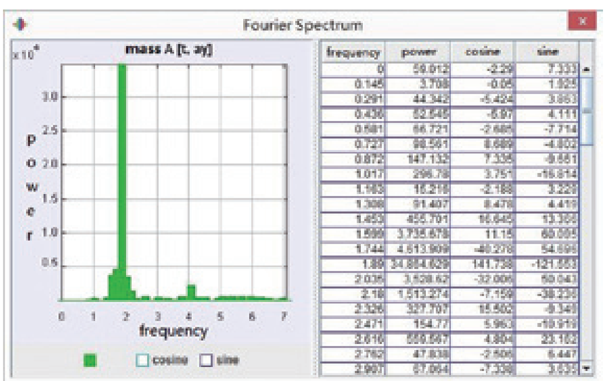


Fig.4. FFT result in x direction

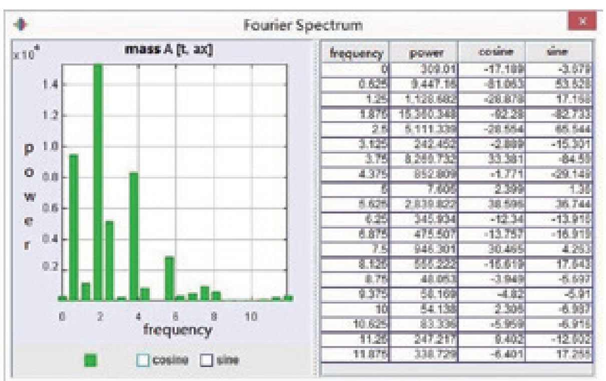


Fig.5. FFT result in y direction

Fig.6. phenomenon at 1.9 Hz

Fig.7. phenomenon of resonance

observed where the peak of the liquid was and shot videos by high-speed camera. Figure 8 shows the shape of the cups and the parameters.

Figure 9 shows the phenomenon of the oscillating liquid in a cup with square cross-section. The peak of water is on the edge so the liquid is more possible for water to spill out along the edge. Figure 10 shows the phenomenon of the oscillating liquid in a cup with excurvature brim and the liquid tends to spill out. Figure 11 also shows the phenomenon of liquid in a beaker with excuvature brim and the liquid is more likely to spill out along the brim. Figure 10 shows the phenomenon of the oscillating liquid in a goblet. Goblet is incurve at the brim and when the liquid oscillates with large amplitude, because of the incurvation at the brim, the liquid tend to turn back to the goblet which makes it safer to walk holding a filled goblet.

Consequently, the goblet-shaped cup is much safer.

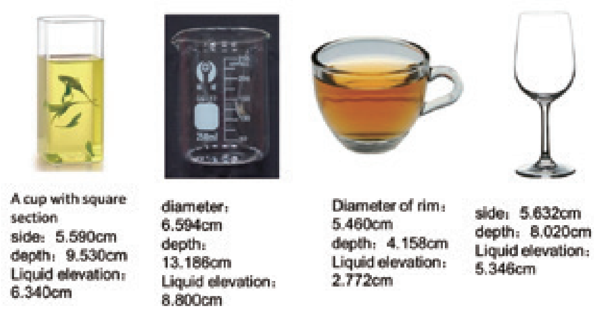


Fig.8. the shape of the cups and their parameters

Fig.9

Fig.10

Fig.11

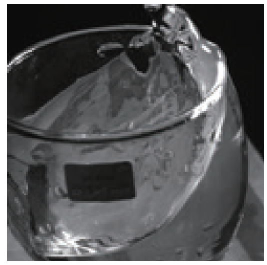


Fig.12

D. Improvement of the cup

In the daily life, we noticed that the tea-strainer in cup can prevent water from spilling out. What if the coffee cup has a tea-strainer? Inspired by this, we did a series of experiments. Five experiments are carried out:

According to the experiment above, we suggest that we can add a small block to the coffee cup like the picture shows below. Only a small block can make a great difference.

Summary

According to the research, following results are given out:

The shape of cup will have a great influence on the possibility to spilt. We recommend using the cups which are resembled to the goblets. Cylindrical cup is a nice choice as well. Cups which are used in café to

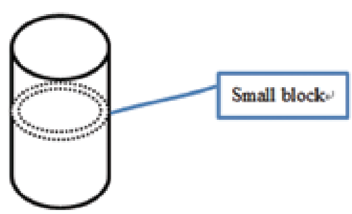
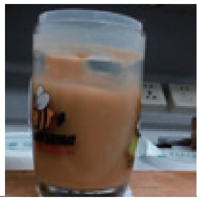
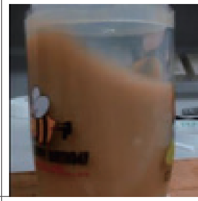
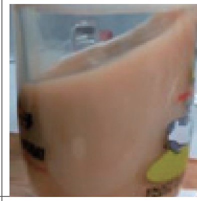


Fig.13. Model of the improved cup

Table 2. Experiments and their results

	Experiment 1	Experiment 2	Experiment 3	Experiment 4	Experiment 5
Tea-strainer				Only a circle left	No tea-strainer
Description	A complete tea-strainer	A cut tea-strainer	Only the edge left	Only a circle left	No tea-strainer
Phenomenon					
Description	The tea-strainer can present the liquid spilling.	Even the block area is decreased, it won't influence much on the effect	When the liquid "jumps" high, the edge can rebound it back and prevent it spilling out.	Same as the phenomenon that there isn't a tea-strainer	Coffee tends to spill out. The amplitude is large and there is a rotation.

latte are not recommended because they tend to spill the liquid out. Cups with square cross section are not recommended either because the liquid tends to spill along the edge of cups. We can also put a small block to prevent it from spilling out.

The rotation of the liquid will decrease the possibility to spill.

The vertical vibration contributes little to the spilt of liquid. The horizontal oscillation is crucial in coffee splash.

Adding a small block helps a lot.

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