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2011 Problem 6 : Convection

Rotating Rayleigh Bernard Convection

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Abstract

Many fluid flows in nature are governed by buoyancy-driven convection. One such classical convection theory, Rayleigh-Bénard Convection (RBC), has demonstrated great success in predicting storms and atmospheric current flows. However, a variety of other natural phenomena, such as geomagnetism, convection in the Arctic Oceans, and behaviors in the interiors of gaseous celestial bodies, are instead governed by its rotating counterpart, *rotating Rayleigh-Bénard Convection (rRBC)*, which is a non-linear phenomenon where the traditional RBC is observed on a rotating frame of reference. rRBC is a non-trivial problem worthy of investigation due to its resemblance to many non-linear systems, spanning climatology, oceanography and astrophysics. Stable states of a rRBC fluid and their evolution have been treated much in depth by current literature, both experimentally and theoretically. However, a linear stability analysis, instead of the traditional variational approach proves effective in locating the critical Rayleigh number beyond which convection sets in. This paper establishes a mathematical model describing this boundary of stable state using a multi-scale perturbative (MSP) approach.

Keywords

Rayleigh-Bénard Convection, rotating frame of reference, multi-scale perturbation.

Introduction

Convection is a mechanism whereby hot fluid rises and cold fluid sinks which gives rises to efficient heat transport between the upper surface of the liquid and the lower one. Rayleigh-Bénard convection (RBC) characterizes the buoyancy-driven nature of this phenomenon and surface-tension-driven convection is thus not considered. The control parameter Rayleigh number [1] as shown in equation (1), indicates the magnitude of thermal driving force of the system,

$$R = \frac{\alpha \Delta T g d^3}{\nu \kappa} \quad (1)$$

with the numerator measuring the order of the driven forces, where α is the thermal expansion rate of the working fluid, g is the gravitational acceleration, ΔT is the temperature difference between the upper and lower plates and d is the height of the convection cell. The denominator denotes the effects of dissipation, with ν being the kinetic viscosity and κ the thermal conductivity of the working fluid. Rayleigh number has a positive correlation with Nusselt number which describes how dominant convection is to conduction. It is within our expectation that a larger Rayleigh number implies a larger likelihood of extensive convection setting in.

There have been numerous research conducted on Rayleigh-Bénard convection in non-rotating cases[2]. These papers mainly focus on describing the critical condition for transition to occur from conduction state, or the basic state, to the convection or spatio-temporal chaos state[3]. Some research papers in the earlier days also tried to derive the basic equation of velocity and temperature fields when the Rayleigh number is further increased from the basic state by the use of perturbation theories. These include the well-known Lorenz attractor which describes the trajectories of fluid particles in a two-dimensional convection cell[4].

Two important features in non-rotating RBC are worth paying attention. Firstly, a critical Rayleigh number exists beyond which convection starts to dominate the heat transport. Secondly, certain patterns can be observed in the non-rotating Rayleigh-Bénard cell. The most notable one is the parallel roll structure formed by the circulating of rising and sinking fluids[5].

However, a variety of other natural phenomena, such as geomagnetism, convection in the Arctic Oceans[6], and behaviours in the interiors of gaseous celestial bodies, are instead governed by its rotational counterpart, *rotational Rayleigh-Bénard Convection (rRBC), which is* a non-linear phenomenon where the traditional RBC is observed on a rotating frame of reference. rRBC is a non-trivial problem worthy of investigation. Fig. 1 shows a typical rRBC setup whereby fluid, heated from below and cooled from the top, spins about its vertical axis with angular frequency ω.

The main approach to calculating the critical condition for linear instability to set in is the variational approach. In this report, the relationship between the critical Rayleigh number and rate of rotation of the system is investigated by the linear stability analysis (LSA) approach, in analogous to the stationary RBC.

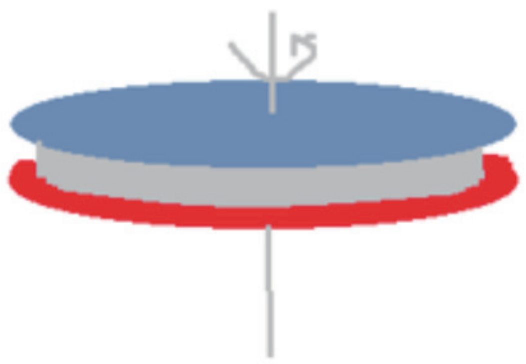

Fig. 1: An illustration of rRBC set-up. Fluid is being heated from below and cooled from above, while the container rotates about its vertical axis.

Boussinesq Equations

Fluid flow of a Newtonian fluid is described by Boussinesq equations[1],

$$\sigma^{-1}(\partial_t + \boldsymbol{u}\cdot\nabla)\boldsymbol{u}$$

$$= -\nabla p + \nabla^2 \boldsymbol{u} + \theta\hat{z} + 2\boldsymbol{\Omega u}\times\hat{z} - \frac{\beta\sigma\Omega}{R_C}\theta\hat{r} \qquad (2)$$

$$(\partial_t + \boldsymbol{u}\cdot\nabla)\theta = \nabla^2\theta + R_w \qquad (3)$$

$$\nabla\cdot\boldsymbol{u} = 0 \qquad (4)$$

The non-dimensionalized constants appeared in the above equations relate to the real physical constants in the following way[5],

$$\sigma = \frac{\nu}{\kappa} \qquad (5)$$

$$\sigma = \frac{\nu}{\kappa} \qquad (6)$$

$$\Omega = \frac{\omega d^2}{\nu} \qquad (7)$$

ω is the angular frequency of rotation. The control parameter r in equation (8), is defined as the amount of deviation from the critical Rayleigh number R_c, beyond which the system becomes unstable.

$$r = \frac{R - R_c}{R_c} \qquad (8)$$

This set of equations contains the non-linear term u·∇u and consequently, we cannot solve for the analytical solution by a linear combination approach. The basic state is given by u=0, which have been chosen to be the conduction-only state where no fluid flow is allowed.

Theoretical Model of Critical Rayleigh Number in rRBC

The Multiple Scale Analysis (MSA) method[7] takes into consideration the time evolution of the perturbation, which may become a secular term in a large time scale, by modulating the slow variables with the fast variables. Based on the symmetry of the system, the following scaling is made[7].

$$\partial_x = \partial_x + r^{\frac{1}{2}}\partial_X \qquad (9)$$

$$\partial_y = r^{\frac{1}{4}}\partial_Y \qquad (10)$$

$$\partial_t = r\partial_T + r^{\frac{5}{4}}\partial_{T'} \qquad (11)$$

And the vector $\boldsymbol{V} = (\boldsymbol{u},\theta,p)$ expands as

$$V = \sum_{n=0}^{\infty} r^{\frac{1}{2}+\frac{1}{4}n} V_n$$

Linear Stability Analysis

Upon introducing the scaled variables given by equation (11), the centrifugal term does not show its effect at lower order of the control parameter r. To our best interest, by setting the centrifugal force term to zero and taking the curl of the momentum equation and the energy equation respectively to eliminate the pressure term:

$$(\nabla^2 - \sigma^{-1}\partial_t)(\nabla\times\boldsymbol{u}) + \partial_y\theta\hat{x} - \partial_x\hat{y} + 2\boldsymbol{\Omega}\partial_z\boldsymbol{u}$$
$$= \sigma^{-1}\nabla\times(\boldsymbol{u}\cdot\boldsymbol{u}\nabla) \qquad (13)$$

$$(\nabla^2 - \partial_t)\theta + \boldsymbol{R}w = \boldsymbol{u}\cdot\nabla\theta \qquad (14)$$

Let $\phi = (u, v, w, \theta) = (u, \theta)$

Equations (2), (11) and (13) can then be written as,

$\boldsymbol{L\phi} = \boldsymbol{N\phi}$

$$L = \begin{pmatrix} 2\Omega\partial_z & -D\partial_z & D\partial_y & \partial_y \\ D\partial_z & D\partial_z & -D\partial_x & D\partial_x \\ -D\partial_y & D\partial_x & 2\Omega\partial_z & 0 \\ 0 & 0 & 0 & \nabla^2 - \partial_t \end{pmatrix} \qquad (15)$$

$$\boldsymbol{N} = \boldsymbol{\sigma}^{-1}\begin{pmatrix} 0 & -\partial_z(u\cdot\nabla) & -\partial_y(u\cdot\nabla) & 0 \\ \partial_z(u\cdot\nabla) & 0 & -\partial_x(u\cdot\nabla) & 0 \\ -\partial_y(u\cdot\nabla) & \partial_x(u\cdot\nabla) & 0 & 0 \\ 0 & 0 & 0 & \boldsymbol{\sigma}(u\cdot\nabla) \end{pmatrix} \qquad (16)$$

Differentiate the first row w.r.t y gives:

$$2\Omega\partial_y\partial_z u - D\partial_y\partial_z v + D\partial_y^{\,2} w + \partial_y^{\,2}\theta$$
$$= \sigma^{-1}\left(\partial_y^{\,2}(u\cdot\nabla w) - \partial_y\partial_z(u\cdot\nabla v)\right) \qquad (17)$$

Differentiate the second row w.r.t x gives:

$$D\partial_x\partial_z u + 2\Omega\partial_x\partial_z v - D\partial_x^{\,2} w - \partial_x^{\,2}\theta$$
$$= \sigma^{-1}\left(\partial_x\partial_z(u\cdot\nabla u) - \partial_x^{\,2}(u\cdot\nabla w)\right) \qquad (18)$$

By taking the difference of equation (14) and (17), and by defining $\nabla_\perp = \partial_x + \partial_y$ for equation (2), we have:

$$2\Omega\partial_z(\partial_y u - \partial_x v) + D\partial_z^{\,2} w + D\nabla_\perp^2 + \nabla_\perp^{\,2}\theta$$
$$= \sigma^{-1}\left(\nabla_\perp^2(u\cdot\nabla w) - \partial_z\nabla_\perp(u\cdot\nabla u)\right) \qquad (19)$$

From the third row and by defining $\partial_x v - \partial_y u = \zeta$, we have:

$$D\zeta + 2\Omega\partial_z w = \sigma^{-1}(\partial_x u \cdot \nabla v - \partial_y u \cdot \nabla v) \qquad (20)$$

Assuming there are two functions ϕ,ψ, such that

$$u = \partial_z\partial_x\phi - \partial_z\psi$$

$$v = \partial_z\partial_y\phi + \partial_x\psi$$

$$w = -(\partial_x^2 + \partial_y^2)\phi$$

We then have $\partial_x v - \partial_y u = \partial_x\partial_y\partial_z\phi + \partial_x^2\psi - \partial_x\partial_y\partial_z\phi + \partial_y^2\psi = (\partial_x^2 + \partial_y^2)$.

Let $x=(\phi,\psi,\theta)$, equations (18) and (19) together with the fourth row give the linear operator:

$$L' = \begin{pmatrix} -D\nabla^2\nabla_\perp^2 & -2\Omega\partial_z\nabla_\perp^2 & \nabla_\perp^2 \\ -2\Omega\partial_z\nabla_\perp^2 & D\nabla^2 & 0 \\ 0 & 0 & \nabla^2 - \partial_t \end{pmatrix}$$

Combined with the linearized energy equation $\partial_t\theta = Rw = R\nabla^2\phi$, solve the steady-state equation $L'\chi = 0$ at order $r^{1/2}$:

$$\nabla^4\nabla_\perp^2\phi + 2\Omega\partial_z\nabla_\perp^2\phi - \nabla_\perp^2\theta = 0 \qquad (22)$$

$$2\Omega\partial_z\nabla_\perp^2\phi - D\nabla_\perp^2\psi = 0 \qquad (23)$$

$$\nabla^2\theta - R\nabla_\perp^2\phi = 0 \qquad (24)$$

Take ∇^2 of equation (20) and add to equation (22) after equation (22) is acted upon by $2\Omega\partial_z\nabla_\perp^2$:

$$(\nabla^6\theta + 4\Omega^2\partial_Z^2\nabla^2\phi - \nabla^2\theta) = 0 \qquad (25)$$

To eliminate $\nabla_\perp^2$, we use the following trial solution, which is the most phenomenological choice based on the roll solution given in the mathematical model corresponding to non-rotating RBC.

$$\theta = \Theta(z)e^{i\,\mathbf{k}\cdot\mathbf{x}_\perp}$$

$$\phi = \Phi(z)e^{i\,\mathbf{k}\cdot\mathbf{x}_\perp}$$

$$\psi = \psi(z)e^{i\,\mathbf{k}\cdot\mathbf{x}_\perp}$$

where $\mathbf{x}_\perp = x\hat{\mathbf{x}} + y\hat{y}$ is the spatial unit vector of the horizontal plane, and $\mathbf{k}$ is the wave-vector of the convection rolls. This gives the following transformation,

$$\nabla^2 = \partial_z^2 + \nabla_\perp^2 = \partial_z^2 + (i\boldsymbol{k})^2 = \partial_z^2 - \boldsymbol{k}^2 \qquad (26)$$

$$\nabla_\perp^2 = -\boldsymbol{k}^2 \qquad (27)$$

Equations (22) to (24) give:

$$\nabla^6\Phi + 4\Omega^2\partial_z\nabla^2\Phi - \nabla^2\Theta = 0 \qquad (28)$$

$$\nabla^2\Theta + \boldsymbol{k}^2\boldsymbol{R}\Phi = 0 \qquad (29)$$

$$2\Omega\partial_z\Phi - \nabla^2\psi = 0 \qquad (30)$$

Addition of equation (27) and equation (28) yields:

$$\nabla^6\Phi + 4\Omega^2\partial_z\nabla^2\Phi + \boldsymbol{k}^2\boldsymbol{R}\Phi = 0 \qquad (31)$$

Substitute in $\nabla^2 = \partial_z^2 - \boldsymbol{k}^2$

$$(\partial_z^2 - \boldsymbol{k}^2)^3\Phi + 4\Omega^2\partial_z^2\Phi + \boldsymbol{k}^2\boldsymbol{R}\Phi = 0 \qquad (32)$$

Using trial solution[8],

$$\Theta(z) = \Sigma_{j=0}^3 C_j \frac{\cosh\lambda_j z}{\cosh\frac{\lambda_j}{2}} \qquad (33)$$

$$\Phi(z) = \sum_{j=0}^{3} C_j\left(\frac{k^2 - \lambda_j^2}{Rk^2}\right)\frac{\cosh\lambda_j z}{\cosh\frac{\lambda_j}{2}} \qquad (34)$$

$$\Psi(z) = \sum_{j=0}^{3} C_j\left(\frac{Rk^2 + (\lambda_j^2 - k^2)^3}{2\Omega Rk^2\lambda_j}\right)\frac{\sinh\lambda_j z}{\cosh\frac{\lambda_j}{2}} \qquad (35)$$

$$(\lambda_j^2 - \boldsymbol{k}^2)^4 + 4\Omega^2\lambda_j^2(\lambda_j^2 - \boldsymbol{k}^2)^2 + RK^2(\lambda_j^2 - \boldsymbol{k}^2) = 0 \qquad (36)$$

Rearranging

$$(\lambda_j^2 - \boldsymbol{k}^2)\left((\lambda_j^2 - \boldsymbol{k}^2)^3 + 4\Omega^2\lambda_j^2(\lambda_j^2 - \boldsymbol{k}^2) + RK^2\right) = 0 \qquad (37)$$

This quadratic equation w.r.t λ_j^2 produces four solutions, but due to the symmetry properties of the proposed trial solution, only λ_j that keeps trial solutions even functions will be chosen.

$$\lambda_0 = k \qquad (38)$$

$$\lambda_1 = \sqrt{k^2 + \frac{\Omega^2}{2\beta} - \frac{8}{3}\beta} \qquad (39)$$

$$\lambda_2 = \sqrt{\frac{\frac{24}{8}(i\sqrt{3}-1)\Omega^2 + 12\beta k^2 + \sqrt[3]{4}\left(i\sqrt[3]{3}+1\right)\beta^2}{12\beta}} \qquad (40)$$

$$\lambda_3 = \sqrt{\frac{\frac{24}{8}(-i\sqrt{3}-1)\Omega^2 + 12\beta k^2 + \sqrt[3]{4}(-i\sqrt[3]{3}+1)\beta^2}{12\beta}}$$

where

$$\beta = \sqrt[3]{27k^2(4\Omega^2 + R) + \sqrt{6912\Omega^6 + 729k^4(4\Omega^2 + R)^2}}$$

Using the boundary conditions, u=v=w=θ=0 at z=±½.

We obtain $\phi = \psi = \theta = \partial_z\phi = 0$ at $z = \pm 1/2$.

Proposed solutions of equations (32) to (34) can be written in matrix form, $Mc = 0$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ \frac{\gamma_0}{\lambda_0} & \frac{\beta_1\gamma_1}{\lambda_1} & \frac{\beta_2\gamma_2}{\lambda_2} & \frac{\beta_3\gamma_3}{\lambda_3} \\ 0 & \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & \gamma_1\alpha_1\lambda_1 & \gamma_2\alpha_2\lambda_2 & \gamma_3\alpha_3\lambda_3 \end{pmatrix}\begin{pmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \end{pmatrix} = 0 \qquad (42)$$

where $\gamma_j = \tanh\frac{\lambda_j}{2}$, $\alpha_j = k^2 - \lambda_j^2$, and $\beta_j = 1 - \frac{\alpha^3}{Rk^2}$

Non-trivial solution leads to zero-valued determinant of the coefficient matrix which produces an equation involving two variables, $\boldsymbol{R}_C$ and Ω.

Implicitly differentiate the equation w.r.t Ω yields $\frac{\partial k}{\partial R}$ which gives the critical control parameter $\boldsymbol{R}_C$ w.r.t the non-dimensionalized rotation rate Ω. A graph of $\boldsymbol{R}_C$ against Ω is presented as depicted in Fig. 2a.

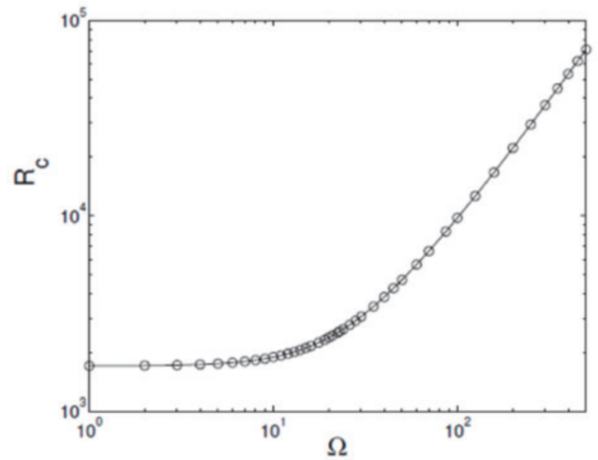

Fig. 2a: A plot of critical Rayleigh number against the scaled rotational rate.

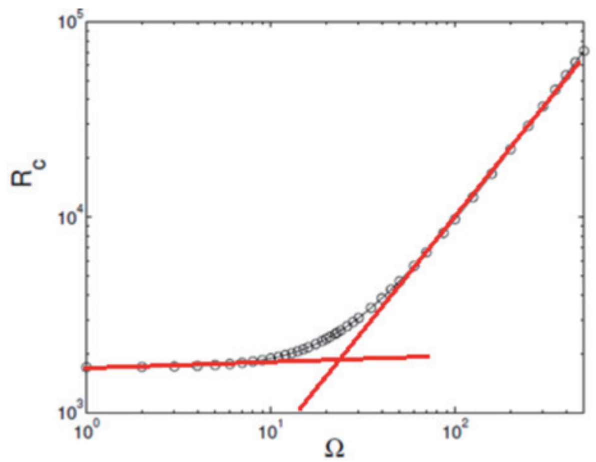

Fig. 2b: Same plot, with linear fits for regions of low and high rotation rates.

Discussion

The critical Rayleigh number, as shown in Fig. 2a, can be approximated to two linear graphs of different gradients at low and at high values of rotation rate, as shown by the 2 line fittings in Fig. 2b. The two linear fits obtained have constants of proportionality of 11 and 170 respectively. At high value of rotation rate, as illustrated in Fig. 2b, the critical Rayleigh number grows with rotation rate by an order of magnitude of 11 as calculated from the gradient.

At low values of rotation rates, the critical Rayleigh number asymptotically approaches 2000 which, in order of magnitude, is equal to its non-rotating counterpart. The critical Rayleigh number in non-rotating RBC is analytically given by $\frac{27}{4}\pi^4 \approx 2065$[2]. Results derived from the Linear Stability Analysis coincide with those derived via variational formulation [2]. The critical Rayleigh number shows a positive correlation with the reduced rate of rotation.

Bajaj et al. [9] also performed experiments investigating the critical behavior of a 50% gaseous mixture of Hydrogen and Xenon and the experimental data supports our above theoretical prediction to a large extent. The experimental results from Bajaj et al. [9] showed that there is also a similar trend to our theoretical graphs obtained, and suggests that our linear stability analysis using a multi-scale perturbative approach proves effective in locating the critical Rayleigh number beyond which convection sets in. The deviation between results from Bajaj et al. and ours can be attributed to the separation of the composition of the mixture, which results due to the coupling between the concentration and temperature fields, observed via the Soret effect [9].

Conclusion

Linear stability analysis does indeed provide a viable way to estimate the critical value of the control parameter and it fits well with preceding experimental and theoretical data. The multiple-scale analysis reduces the non-linear problem to a linear problem, eliminates the second-order manifestation of the rotating frame and thus simplifies the problem. An analogy between the rRBC and stationary RBC can then be drawn, simplifying the calculation process. Nevertheless, a linear stability analysis leaves a solution that grows exponentially with time due to the assumption of choosing null space of $\boldsymbol{L'}$ ($\boldsymbol{L'}\chi=0$), and can only be fixed by the correction from higher order expansion of control parameter, r.

We chose to ignore the effect of centrifugal force in the Linear Stability Analysis due to its minimal contribution at low order of ε. Further studies in higher order terms can lead to more interesting results of convection dynamics such as spiral and chiral symmetry breaking[10]and spiral defect chaos. At even higher values of the control parameter, a statistical means may prove more effective in predicting flow fields and pattern formation, which could be the subject of future studies.

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