2017 Problem 2: Balloon Airhorn

A Novel Mathematical Approach

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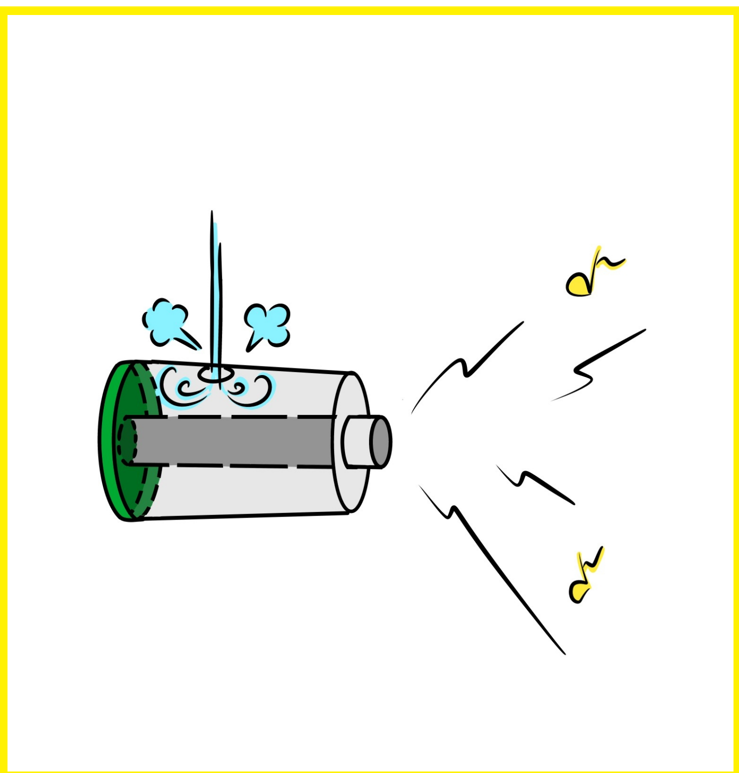

Abstract

The Balloon Airhorn is a problem of the 30th International Young Physicists' Tournament. This study proposes a qualitative theory explaining the mechanism of sound production of the airhorn, and a mathematical theory modelling the sound production fo the horn. An airhorn was constructed as per the instructions detailed in the IYPT problem, and the frequency and intensity of the sound produced was investigated while varying the length of the container, the length of the tube and the protrusion length of the tube against the membrane of the airhorn. A close correlation between the predicted trends of our mathematical theory and experimental results were obtained. Based on the mathematical model of the balloon airhorn system, a computer problem to predict the frequency of the sound produced by the airhorn was also written in the Wolfram Mathematica language.

1 Background and purpose

A simple airhorn can be constructed by stretching a balloon over the opening of a small container or cup with a tube through the other end. Blowing through a small hole in the side of the container can produce a sound. The principle of a balloon airhorn is similar to that of a conventional airhorn, with a rubber membrane (the "balloon") replacing the vibrating reed as the source of sound production. The airhorn has found use in signalling purposes such as on ships, and civil defence sirens for emergency situations. A balloon airhorn exhibits desirable characteristics compared to a conventional airhorn with a vibrating reed, being simpler in design, and that the dimensions of the inner tube can be changed to alter the sound produced.

However, to the author's knowledge there has been no comprehensive study on the engineering mechanics behind a balloon airhorn, and how changing the parameters of the airhorn would affect the sound produced. Due to a lack of a mathematical theory to describe the behaviour of a membrane-driven horn system and a lack of understanding into the factors affecting the behaviour of

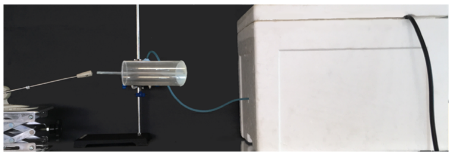

FIG. 1. Picture of Experimental Setup Used

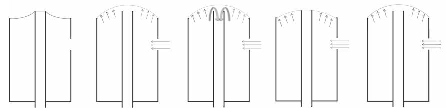

FIG. 2. Illustrations of the oscillation of the membrane from rest

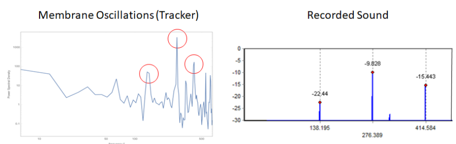

FIG. 3. Fourier transformed plot of membrane oscillation (left, peaks highlighted in red) and the sound produced by the airhorn (right)

such a system, there is insufficient information to allow the application of such a signalling system for civilian and defence applications.

Hence, this study aims to develop a mathematical approach to modelling the balloon airhorn system, and collect experimental results to characterise how the height of tube protruding against the membrane, length of the tube and length of the container would affect the intensity and frequency of the sound produced from the balloon airhorn. Such information would be of use to engineers considering the application of a balloon airhorn or related sound-producing system in signalling applications.

2 Hypotheses

The increase of the height of tube protruding against the membrane will cause an increase in sound frequency and a decrease in sound intensity of the airhorn; the increase of the length of the tube will cause a decrease in sound frequency but the sound intensity of the airhorn will stay constant; the increase of the length of the container will cause an increase in sound frequency and a decrease in sound intensity of the airhorn.

3 Methodology

A balloon airhorn was constructed from acrylic tubes of varying dimensions, with a rubber membrane affixed to one end. Air was fed into the airhorn using an air blower insulated in a Styrofoam box containing acoustic foam. A microphone placed at a fix distance of 15 cm from the opening of the airhorn was connected to a laptop to record the sound produced by the horn. The sound was then analysed by Fast Fourier Transform to obtain the constituent frequencies. The main peak on the Fourier transform plot was taken to be the main resonant frequency of the airhorn, while the amplitude of the main peak was taken to be the intensity of the sound produced by the airhorn.

To characterise membrane oscillation, a high-speed camera was used to record the movement of the membrane at 2000 frames per second, and the subsequent high-speed video was then analysed using the video processing software Tracker.

4 Theory

4.1 Qualitative Theory of Sound Production

When there is no air input, the membrane lies in a rest state and no air is able to enter the tube. Air input causes the pressure in the container to increase, and the membrane takes the shape of a spherical cap due to equal pressure at every point on the membrane. Air rushes into the tube from the pressurised container, causing the pressure in the container to rapidly decrease. The membrane then collapses, blocking air from entering the tube. Pressure in the container increases again, and process of collapsing and expansion of the membrane repeats. This membrane oscillation causes air flow rate out of the tube to vary, causing the airhorn to produce the sound.

This qualitative theory of airhorn sound production suggests that the frequency fo membrane oscillation would be equal to the frequency of the sound produced. Comparing the Fourier transformed data of membrane oscillation and that of the recorded sound, the main peaks are at similar frequencies (137 Hz, 274 Hz, 411 Hz for membrane oscillation, and 138 Hz, 276 Hz, 415 Hz for recorded sound). This lends support to the qualitative described above.

4.2 Quantitative Theory of Sound Production

Key parameters:
x = displacement of membrane, h = protrusion length of tube, L = length of tube, R = radius of component, Q = volumetric flow rate of blower, $P[t]$ = pressure in the cup

Of these, three variables are unknown: the displacement fo the membrane, the pressure in the

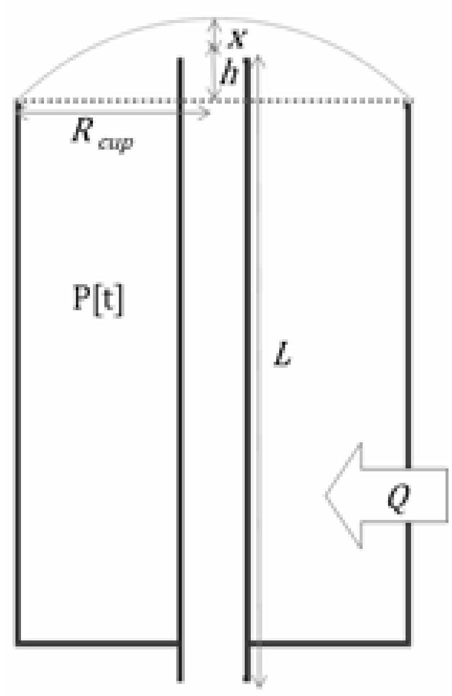

FIG. 4. Drawing of cross-section of the airhorn, with key parameters labelled

container and the molar flow rate out of the system through the tube. Three equations describing the system are required to solve for the unknown variables.

4.2.1 Understanding membrane oscillation

Fichter et. al describes a mathematical model for a uniformly loaded circular membrane [2]. Considering the case of an infinitesimally small section of the airhorn membrane with applied pressure q, tension σ acting at angle θ from the horizontal, horizontal length r and membrane thickness h, balancing of vertical forces yields us Eq. (1) [2]:

$$2\pi rh\sigma \sin\theta = \pi r^2 q \tag{1}$$

Which can be simplified with the use of the small angle

$$\sin\theta \approx \tan\theta = \frac{dy}{dr} \tag{2}$$

The forces acting on a small radial section of the membrane are shown in FIG. 6, where σ_i is the circumferential stress, and σ_r is the radial stress acting on the membrane. Balancing the radial forces acting on the membrane:

$$drh\sigma_i d\theta = rhd\theta\sigma_r - (r+dr)d\theta h\sigma_{r+dr} \tag{3}$$

Simplifying, we get:

$$h\sigma_i = \frac{d}{dr}\left(rh\sigma_r\right) \tag{4}$$

However, considering that the membrane has a nonzero slope, the normal pressure has a radial component, thus making the complete equation:

$$h\sigma_i = \frac{d}{dr}\left(rh\sigma_r\right) - qr\frac{dw}{dr} \tag{5}$$

The well-known strain equations gives the radial (ε_{radial}) and circumferential (ε_{circum}) strain of a radial section of the membrane defined by the two radial distances u and r ($u > r$) [2]:

$$\varepsilon_{radial} = \frac{du}{dr} + \frac{1}{2}\left(\frac{dy}{dr}\right)^2 \tag{6}$$

$$\varepsilon_{circum} = \frac{u}{r} \tag{7}$$

The stress-strain equations relates stress to strain as a function of the elastic modulus E of the membrane:

$$\sigma_i - \mu\sigma_r = Eh\varepsilon_{circum} \tag{8}$$

$$\sigma_r - \mu\sigma_i = Eh\varepsilon_{raidal} \tag{9}$$

Elastic modulus of the membrane was measured experimentally to be 0.5 MPa. Solving Eq. [(1)-(9)] allows us to characterise the restoring force experienced by the elastic membrane.

At any one time, the acceleration of the membrane depends on the net force experienced, which depends on the force balance between the restoring force of the membrane and the force exerted by the pressure in the container:

$$\frac{d^2x}{dt^2} = \frac{F_{restoring}[x]}{m} - \frac{2\pi R_{cup}^2(P[t]-P_{atm})}{m} \tag{10}$$

From Eq. 10, it can be seen that characterising the oscillation of the membrane requires an understanding of the pressure changes in the container of the airhorn, which will be detailed in the following section.

4.2.2 Understanding pressure changes

Under the conditions experienced in the airhorn system, air can be approximated as an ideal gas:

$$P = \frac{NRT}{V} \tag{11}$$

The amount of air, n, in the airhorn at any one time consists of the air present in the horn before excitation, the molar flow rate into the horn

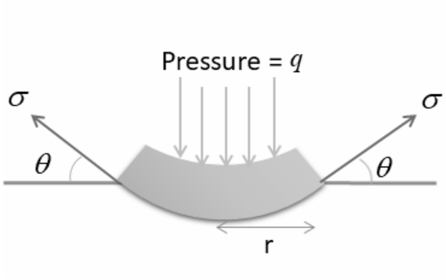

FIG. 5. Cross-section of an infinitesimally small section of the airhorn membrane

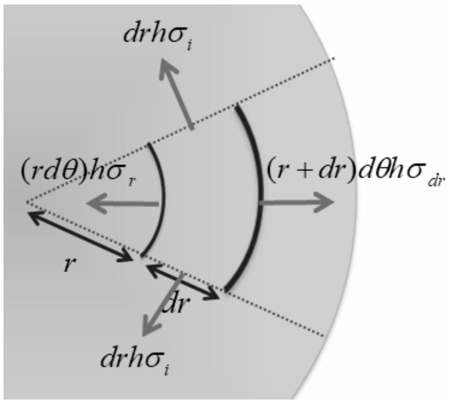

FIG. 6. Top view of a small section of the airhorn membrane

($\acute{M}_{input}$) and the molar flow rate out of the horn ($\acute{M}_{tube}$):

$$n = \frac{V_0 P_{atm}}{RT} + \acute{M}_{input} t - \acute{M}_{tube} \tag{12}$$

Where molar flow rate can be expressed in terms of air density ρ, volumetric flow rate Q, and molar mass of air M:

$$\acute{M} = \frac{\rho Q}{M} \tag{13}$$

From Eq. [(11) - (13)] the pressure in the container can be expressed as a function of molar flow rate into and out of the system, and the change in volume of the airhorn caused by membrane oscillation:

$$P(t) = \frac{RT(n_0 + \acute{M}_{input} t - \acute{M}_{tube} t)}{V_0 + V[x,t]} \tag{14}$$

Air pressure drops from $P(t)$ in the container to P_{atm} outside the airhorn tube. There are three main processes that result in this loss of pressure. Firstly, the pressure loss when air is compressed from the container into the smaller tube; secondly, the pressure loss as air curves into the tube; lastly, the pressure loss as air flows through the tube (frictional loss).

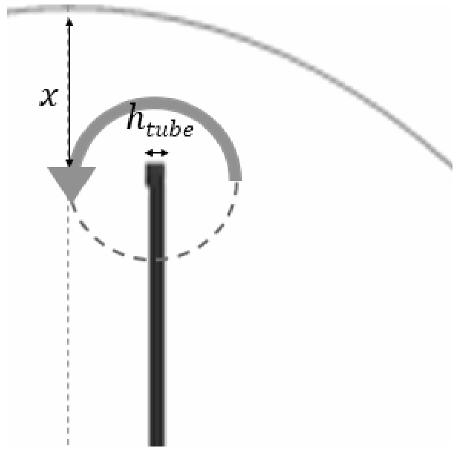

FIG. 7. Simplified representation of air flowing from the container to the tube

To model the pressure loss of the air as it is compressed from the larger container to the smaller tube, air is first modelled by the ideal gas equation:

$$PV = nRT \tag{15}$$

Assuming a constant mass flow rate of the air and consider molar density of air:

$$d(\rho v A) = 0 \tag{16}$$

From conservation of energy,

$$P + \frac{1}{2}\rho v^2 = 0 \tag{17}$$

Differentiating Eq. (17) with respect to air velocity v:

$$dP + \rho v dv = 0 \tag{18}$$

Assuming the enthalpy of the air remains constant:

$$K.E. + Q = 0 \tag{19}$$

Where $K.E.$ is kinetic energy of the air.
Thus we can obtain:

$$C_p dT + v dv = 0 \tag{20}$$

Solving Eq. [(15), (16), (18), (20)], we obtain a relationship between the loss of pressure arising from compression of the air and the molar flow rate.

Ito (19960) empirically determined the pressure loss as air moves through smooth bent tubes total pressure loss depends on the bend loss coefficients (friction factor K_d, dependent on tube diameter d; curvature coefficient K_r, dependent on radius of curvature of the bend; angle coefficient K_a, dependent on bend angle), and is also determined by the cross-sectional area of the bent tube A, the air density ρ, and volumetric flow rate q [2]:

$$\Delta P = (K_d K_r K_a)\frac{\rho}{2A^2}q|q| \qquad (21)$$

As shown from FIG. 7, air flowing from the container into the tube can be modelled as air flowing through two smooth 90° bent tubes, with tube diameter taken to be the distance of the membrane from the tube, $x[t]$. Radius of curvature is taken to be $\frac{x}{2} + h_{tube}$, and Eq. (21) is solved to find the pressure loss caused by air rounding the bend.

To approximate the pressure loss as air flows through the tube, the Hagen-Poiseuille equation can be used. The Hagen-Poiseuille equation is a well-known equation used to model laminar flow in smooth pipes, and can be expressed as:

$$\Delta p = \frac{128\mu L Q}{\pi d^4} \qquad (22)$$

Where μ is the viscosity of air, L is the length of the tube, Q is the volumetric flow rate and d is the diameter of the tube.

Plotting the three sources of pressure loss – pressure loss due to compression from the container to the tube, pressure loss caused by rounding the bend from the container to the tube and pressure loss due to air flowing through the tube – against the height of the membrane above the tube (X) allows us to see the relative magnitudes of the pressure losses (FIG. 8).

4.2.3 Program written for frequency prediction

Using the equations described in the Theory section, the position of the membrane (X) as a function fo time can be solved. A program was written in Wolfram Mathematica to predict the membrane oscillation given the parameters of the airhorn. The graph for membrane oscillation is then Fourier Transformed and the main peak taken to be the frequency of membrane oscillation. The frequency of the sound produced by the airhorn is taken to be the same as the frequency of membrane oscillation. This program is useful as it allows for the comparison of our theoretical model with collected experimental data, and is useful for engineers who wish to obtain an estimated frequency of sound production for a certain balloon airhorn design.

5 Results and discussion

5.1 Frequency of airhorn

From FIG. 9, as the protrusion height of the tube increases, the frequency of the horn increases. This

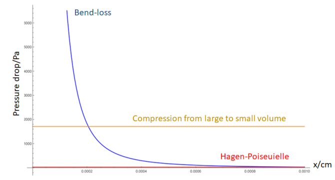

FIG. 8. The three sources of pressure loss plotted against height of the membrane, x

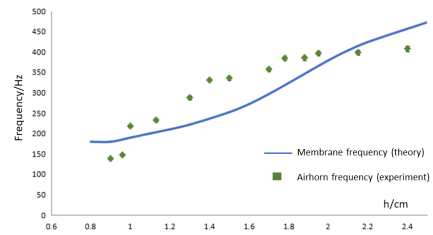

FIG. 9. Frequency of airhorn against height of tube protruding against the membrane (h)

is likely due to tension in the membrane increasing as the protrusion length of the tube pressing against the membrane increases resulting in the membrane oscillating at a higher frequency. As shown from the graph, the theoretical model provides a good estimate of the rough frequency of the airhorn given a certain protrusion height of the airhorn tube.

Our mathematical theory predicts close to no change in airhorn frequency as tube length is varied, due to the small contribution in pressure loss from air flow through the tube (FIG. 8). However, from FIG. 10, as the length of the tube increases, frequency of the airhorn decreases from $L = 15$ cm to $L = 30$ cm, before reaching a plateau at around 270 Hz. As seen from the graph, from $L = 15$ cm to $L = 30$ cm, frequency of the airhorn follows a trend similar to the resonant frequency (fundamental mode) of the tube, following $L = 30$ cm, frequency of the airhorn follows our mathematical model predicting no change to airhorn frequency as length of tube increases. This suggests acoustic coupling between the membrane and the airhorn tube.

As can be seen from FIG. 11, the mathematical theory predicts that as the container length increases, the frequency of the airhorn will increase. However, the measured frequency of the airhorn does not follow the predicted trend, with values between our mathematical theory and the resonant frequency of the airhorn tube. This further suggests a form of acoustic coupling between the airhorn tube and the membrane.

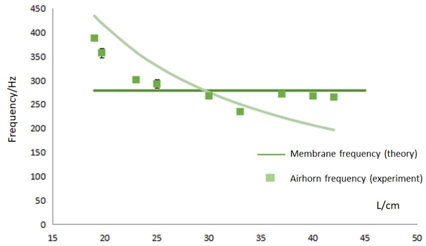

FIG. 10. Frequency of airhorn against length of the tube (L). Light green curve represents resonant frequency of the tube

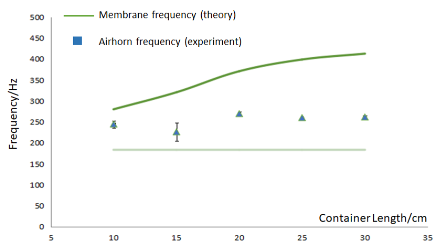

FIG. 11. Frequency of airhorn against length of the container. Light green line represents resonant frequency of the tube

5.2 Sound intensity of airhorn

From FIG. 12, as the protrusion height of the tube increases, the sound intensity stays relatively constant. This is likely due to the sound pressure not depending on the tension of the membrane. As shown from the graph, the theoretical model provides a good estimate of the rough frequency of the airhorn given a certain protrusion height of the airhorn tube.

We predict relatively no change in sound intensity as tube length is varied. However, from FIG. 13, as the length of the tube increases, frequency of the airhorn decreases from $L = 15$ cm to $L = 25$ cm, before increasing from $L = 25$ cm to $L = 43$ cm. As seen from the graph, as length of tube increases, sound detector further away, causing sound intensity to decrease at lower tube length. However, increasing length of tube has greater amplifying effect on sound, overcoming the further distance of the sound detector, leading to the increase of sound intensity at higher tube length.

Our mathematical theory predicts that as the length of the container increases, the sound intensity decreases due to change in volume of container at higher container length has less effect on pressure. The experimental data (FIG. 14) verifies our theory with a corresponding decrease trend.

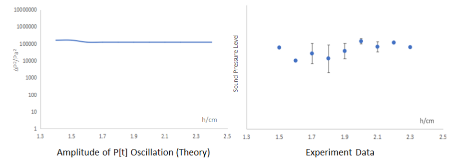

FIG. 12. Sound intensity against height of tube protruding against the membrane (h)

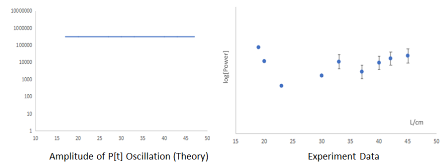

FIG. 13. Sound intensity against length of tube (L)

6 Conclusion and future study

We have presented a new quantitative theory in explaining the sound mechanism of balloon airhorns and have shown that our simulation could give a good estimate of the frequency and the intensity of the sound produced by the airhorn verified by experiments.

It is found that the increase of the height of tube protruding against the membrane will cause an increase in sound frequency, but it has no effect on sound intensity of the airhorn; the increase of the length of the tube will cause a decrease in sound frequency but the sound intensity of the airhorn will stay constant; the increase of the length of the container will cause a decrease in sound intensity of the airhorn but has not effect on sound frequency.

In future study, acoustic coupling between the airhorn tube and the membrane and the distance of sound detector from the vibrating source are needed to be considered in order to increase the accuracy of our theoretical predictions to achieve a better understanding of the behaviour of balloon airhorn for more accurate implementation in civilian and defence signalling system.

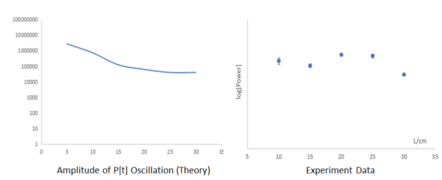

FIG. 14. Sound intensity against length of container

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