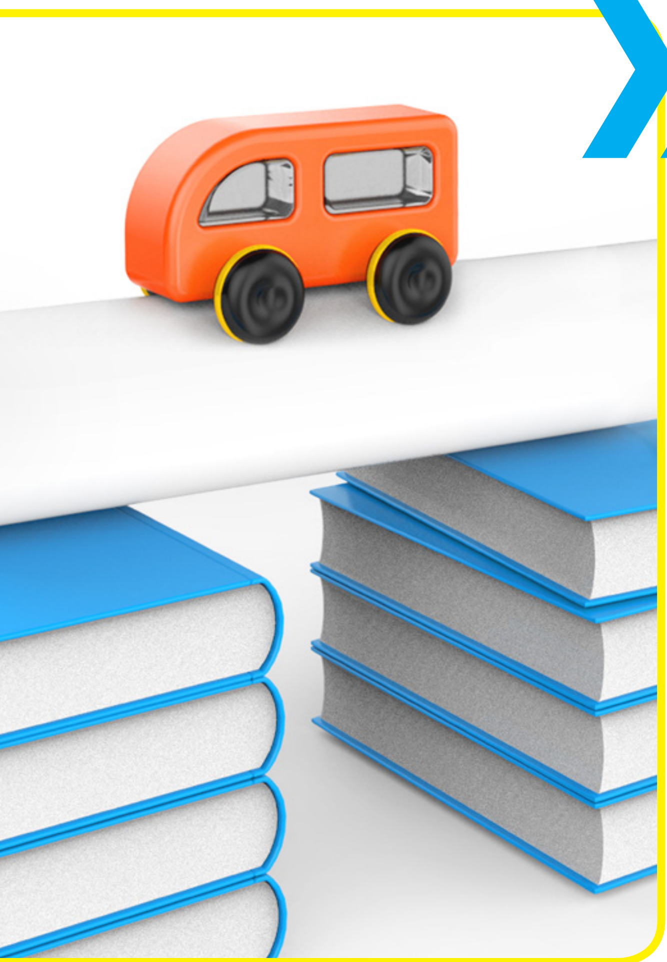

2013 Problem 1 : Invent Yourself

Optimizing the strength of an A4 sheet bridge

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ABSTRACT

An A4 sheet, if it is folded accordion style or rolled into a tube, can withstand much bigger load than a normal A4 sheet, because the second moment of area increases. This research paper first compares the two types of bridge and analyzes the collapse mechanism of an accordion style bridge. Then the relationship between the geometry of a truss (number of bumps, contact angle, and length ratio) and the strength is investigated experimentally.

Keywords

A4 sheet, second moment of area, ultimate compressive strength.

Introduction

Normally, an A4 sheet cannot withstand a weight without being deformed. However, if the paper is fold into certain shape or rolled like a tube, it can stand much bigger weight. There are many preceding studies on the optimization of the paper bridge, yet most of them are not built upon theoretical explanation. Before investigating the phenomenon and optimizing the strength of the paper bridge, this paper first answers why certain paper bridge can be stronger than the others, by analyzing different collapse scenarios. Then pre-experiments are conducted to verify such collapse mechanisms. Finally the optimized truss structure is found empirically. To simplify the problem, the parameters are narrowed down based on the collapse model. The strength of a paper bridge is defined as the maximum mass that a paper can withstand before it collapses completely.

Theory1 - Second moment of area of a paper bridge

The second moment of area of a paper bridge is one of the important factors that determine the strength of the bridge. The distribution of a cross-sectional area of a beam about the axis, perpendicular to the bending direction, is represented by the second moment of area.[1] For example, in a Cartesian coordinate, the second moment of area about x-axis is $I_z=\int\int_A y^2\, dzdy$. In the case of a paper bridge, the horizontal axis, lying across the gap is chosen as x-axis. The second moment of an ordinary A4 sheet is almost zero, yet it is increased if the paper is folded or rolled into a tube.

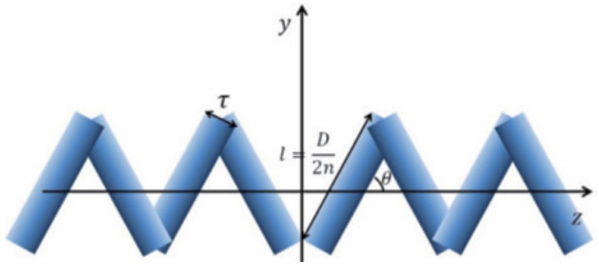

Fig. 1: Cross-section of a truss.

First, the second moment of area of a truss can be derived from a rectangle. A rectangle with width τ and length l is calculated as the following:

$$I_{z0} = \frac{\tau l^3}{12} \tag{1}$$

The cross-section of a truss with $\boldsymbol{n}$ bumps can be treated as $\boldsymbol{2n}$ rectangles in series, which are rotated by $\boldsymbol{\theta}$ from the y axis. If $\boldsymbol{\tau \ll l}$, the second moment of area is expressed as the following equation:

$$I_{z\theta} = \frac{n\tau l^3}{12}(1-\cos 2\theta) \tag{2}$$

Since the length of the rectangle depends on the total length of the A4 sheet and the number of bump, the moment of a truss is determined as the Eqn. 3:

$$I_{z.truss} = \frac{D^3\tau}{96n^2}(1-\cos 2\theta) \tag{3}$$

Here D is the length(210mm) and τ is the thickness(0.14mm) of an A4 sheet.

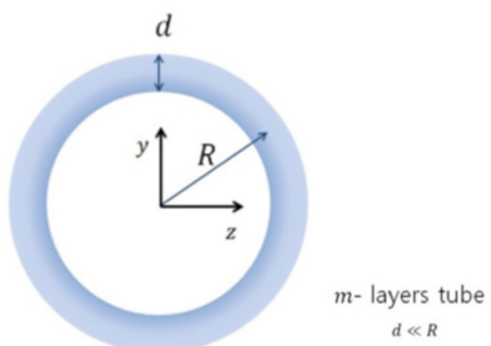

Fig. 2: Cross-section of a tube.

Next, let's find the second moment of area of a tube. The cross section of a tube of m layers is a circle of the radius R and the thickness d as the Fig. 2 shows. Thus, the second moment of area is as the following:

$$I_z = \frac{\pi}{4}\left(\left(R+\frac{d}{2}\right)^4 - \left(R-\frac{d}{2}\right)^4\right) \approx \pi R^3 d \tag{4}$$

R and ***d*** can be substituted by the number of layers, thickness, and the total length of an A4 sheet.

$$\boldsymbol{d = m\tau} \tag{5}$$

$$R = \frac{D}{2\pi m} \tag{6}$$

Therefore, the resulting moment is expressed as the Eqn. 7.

$$I_{z.tube} = \frac{D^3\tau}{8\pi^2 m^2} \tag{7}$$

The range of the contact angle $\boldsymbol{\theta}$ is from 0 to 90°, that of $\boldsymbol{n}$ is from 1 to 7, and $\boldsymbol{m}$ from 1 to 10. The ratio between the two values of second moment of area, the strength of truss and tube can be compared in simplified case – the friction, ultimate compressive strength, etc. are not considered yet.

$$z = \frac{I_{z.truss}}{I_{z.tube}} = \frac{\pi^2(1-\cos 2\theta)m^2}{12n^2} \tag{8}$$

The following Fig. 3 plots the z values, which are higher than 1, in accordance with $\boldsymbol{m/n}$ and $\boldsymbol{\theta}$.

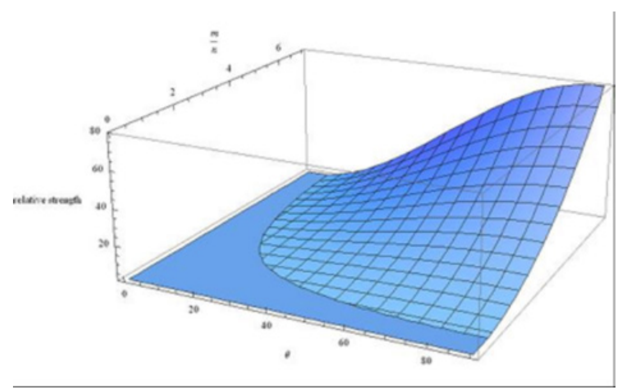

Fig. 3: Ratio of the strength of a truss and a tube, with respect to m, n, and θ

In the Fig. 3, the probability that z exceeds 1 in the given range of n, m, and θ is 0.720: As the contact angle increases, from certain point the second moment of area of a truss exceeds that of a tube. Since the truss becomes stronger than a tube if the angle is big enough, this research focusses on the truss structure. Moreover, because of the complex collapse mechanism, more topics can be investigated about the truss structure.

Theory2-Collapse Mechanisms of a Truss

To find the optimization condition of a truss, its collapsing process must be clarified. The collapse of a truss can be explained by three mechanisms: buckling, sliding, and necking.

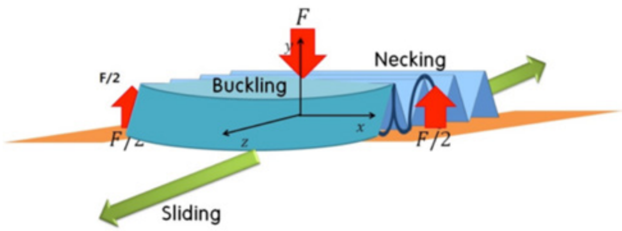

Fig. 4: Collapse mechanisms of a truss: buckling, sliding and necking.

The buckling is the collapse that occurs when a paper bridge is bent in the xy plain. Since a paper sheet has very small compressive and tensile limit, the ultimate strength should be considered to find the collapse condition. Since the cellulose has much smaller ultimate compressive stress than the ultimate tensile stress [2], the collapse starts as the upper part of the bridge is locally deformed. The Fig. 5 is the observed local deformation of a paper bridge.

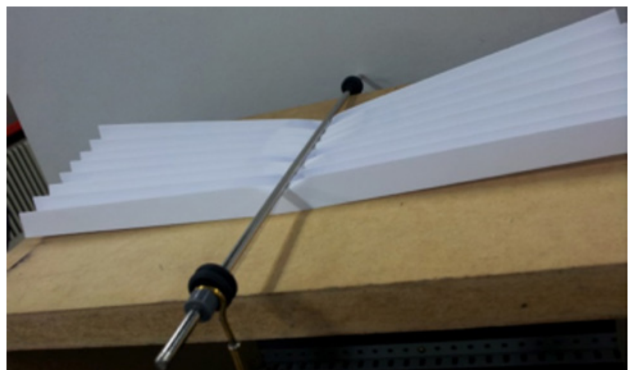

Fig. 5: Local deformation of a truss, due to the ultimate compressive limit.

By calculating the stress at the top of the paper bridge, the strength of a truss is predicted. If a paper bridge is pressed along the z-axis at the middle, the bending moment of the bridge is distributed as the Fig. 6 shows.

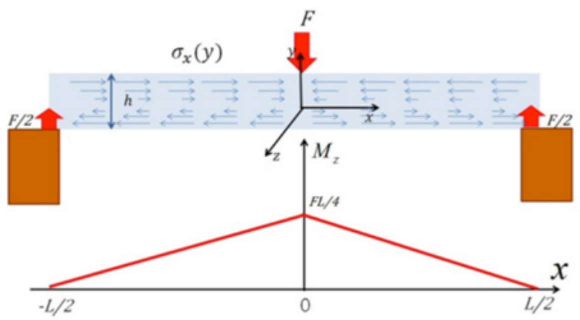

Fig. 6: Distribution of internal stress and bending moment of a truss.

The maximum bending moment at the center is expressed by both the weight on the bridge and the compressive stress at the top.

$$M(0) = \frac{FL}{4} = \frac{\sigma_{top} I}{y} \quad (9)$$

From the Eqn. 14 and the ultimate compressive limit, the maximum force on the bridge before the collapse is found.

$$F_{max} = \frac{4\sigma_{max} I}{Ly} \quad (10)$$

The second moment of area and the height can be expressed with the standard size of an A4 sheet, as mentioned in the Theory 1.

$$y = \frac{h}{2} = \frac{D\sin\theta}{2n} \quad (11)$$

$$I_z = \frac{D^3\tau}{96n^2}(1-\cos 2\theta) \quad (12)$$

The resulting maximum force on the bridge is shown as the Eqn. 13.

$$F_{max} = \frac{\sigma_{max} D^2 \tau \sin\theta}{6nL} \quad (13)$$

The maximum force increases as the contact angle increases and as the number of bump decreases, according to the Eqn. 13. Since σ_{max} of an A4 sheet is in 10GPa scale, the range of strength of a paper truss is predicted as the following:

$$34.6\text{g} \le \frac{m_{max} n}{\sin\theta} \le 346\text{g} \quad (14)$$

The second mechanism, sliding is the unfolding process of the bridge members. It is initiated as the frictional force in z-direction reaches the maximum static friction. Considering only the sliding effect, a bridge member is in torque equilibrium.

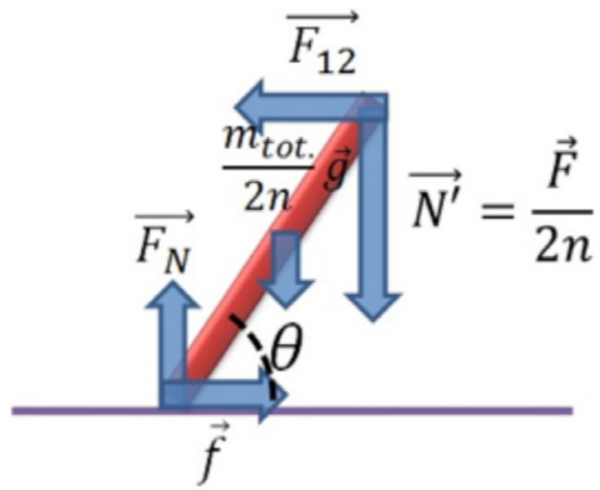

Fig. 7: Free body diagram of a bridge member in equilibrium.

$$\frac{m_{tot.}g}{2n}\cos\theta\frac{l}{2} + \frac{F}{2n}\cos\theta\, l - fl\sin\theta = 0 \quad (15)$$

Thus, in the equilibrium, the frictional force on a bridge member is expressed in terms of the load on the bridge, number of bumps, and contact angle.

$$f = \frac{\cot\theta}{2n}\left(F + \frac{m_{tot.}g}{2}\right) \approx \frac{\cot\theta\, F}{2n} \quad (16)$$

Since the sliding condition is f=μF_N, the threshold angle is expressed as the Eqn. 17.

$$\tan^{-1}\frac{1}{\mu} = \theta \quad (17)$$

If the contact angle becomes smaller than this threshold value, the bridge member starts to accelerate in z direction, and the bridge collapses.

Finally, the necking indicates the deflection of the individual bridge members in the yz plain. The necking occurs when the stress on each bridge member exceeds the ultimate compressive stress. The Fig. 8 shows the force parallel to each bridge member.

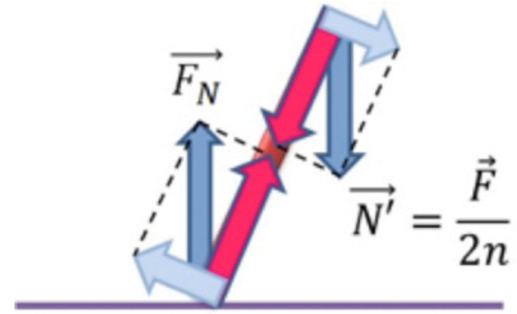

Fig. 8: Stress on each bridge member.

If the length of the part that is on the edge of the gap is L', the stress in a bridge member is shown as the following equation:

$$\sigma = \frac{N'\sin\theta}{A} = \frac{F\sin\theta}{2n\tau L'} \quad (18)$$

The maximum force on the paper bridge is, therefore, shown as the Eqn. 19.

$$F_{max} = \frac{2n\tau L'\sigma_{max}}{\sin\theta} \quad (19)$$

In reality, a paper truss undergoes the combination of the three collapse mechanisms, as the Fig. 9 demonstrates. In fact, a bridge is more weakened by the combined effect of the necking and sliding. As the individual bridge member bends downward, the contact angle θ decreases from the initial value. Therefore, the strength of the bridge, represented by m_(tot.), has to decrease according to the Eqn. 17.

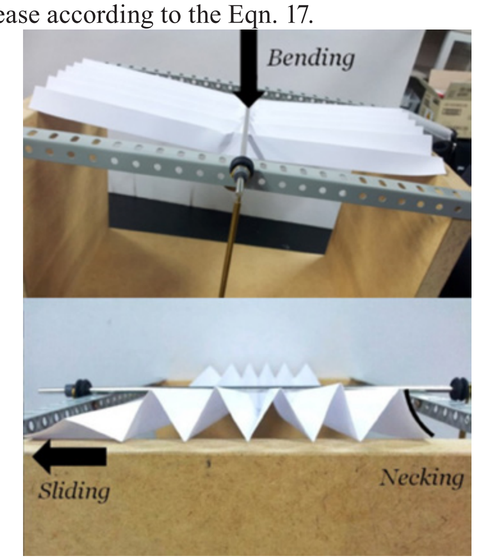

Fig. 9: Three collapse mechanisms combined.

To summarize, considering the buckling and the sliding effect, the strength of a bridge increases as the number of bump decreases and as the contact angle increases. However, considering only the necking effect, the opposite tendency is predicted: as the number of bump increases and as the contact angle decreases, the strength increases.

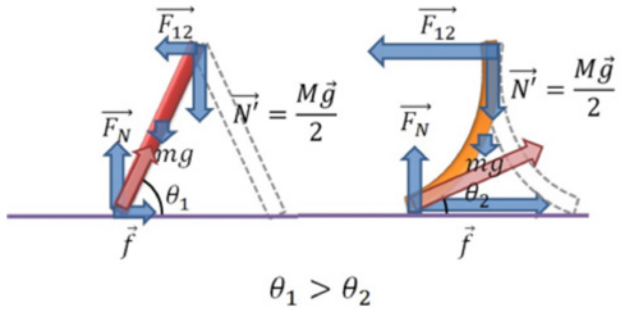

Fig. 10: Necking and sliding effects combined.

Experiment 1 -Necking of a Truss

The Fig. 11 shows a modified truss structure, which is used to verify the collapse mechanism due to the necking effect. This paper bridge is attached to two other A4 sheets at the top and the bottom and placed on the flat ground. The force on the bridge is equally distributed by placing the weights on a hard board. Thus, the other two collapse mechanisms were eliminated.

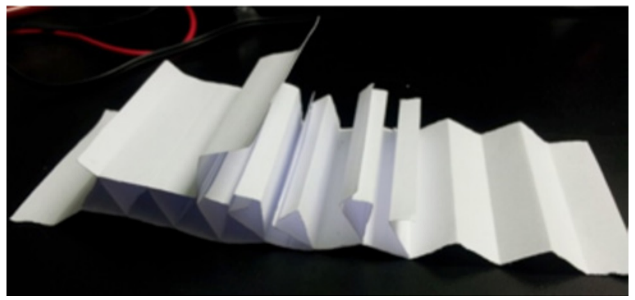

Fig. 11: Each bridge member is attached to two A4 sheets at the top and the bottom.

As the mass increases, the bridge members are bent slightly, and as it reaches ultimate compressive limit, the bridge suddenly collapses.

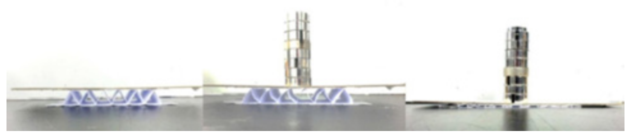

Fig. 12: A truss designed to verify the necking effect. It suddenly collapses at the ultimate compressive limit.

According to the Eqn. 6, the number of bump and the contact angle influence the strength of a truss. First, when the number of bumps increases from 2 to 7 and the contact angle is 60°, the strength of the bridge changes as the graph in the Fig. 13 demonstrates.

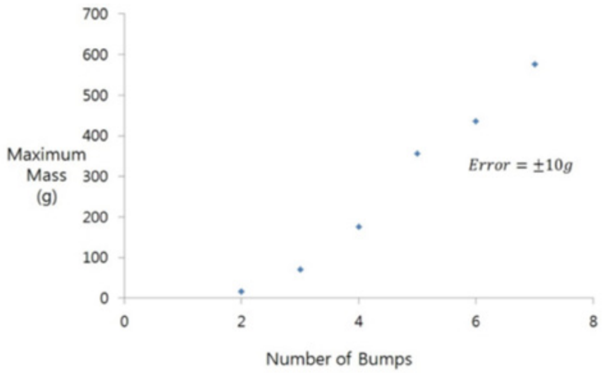

Fig. 13: The strength of the bridge increases as the number of bumps increases, considering only the necking effect.

If n is bigger than 3, the strength of the bridge follows the linear tendency predicted in the Eqn. 19. However, when the number of bumps is small, the mass of the individual bridge members cannot be ignored. Thus, the strength of the bridge, under n=3, is lower.

Next, as the contact angle increases from 40° to 80°, the strength of the bridge decreases as the graph in the Fig. 14 illustrates.

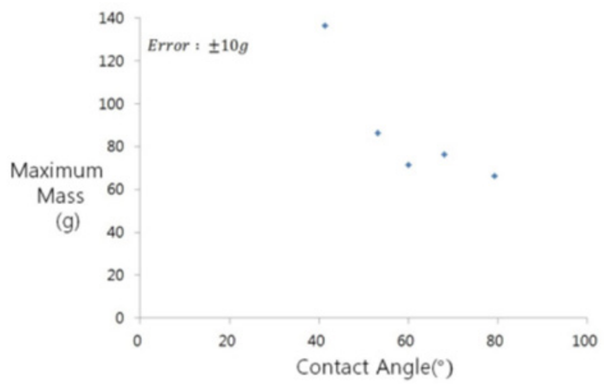

Fig. 14: The strength of the bridge decreases as the contact angle increases, considering only the necking effect.

Thus, the strength of the truss increases as the number of bumps increases and as the contact angle decreases, if only the necking effect is considered.

Experiment 2 – Strength of a Truss and its Optimization

If a paper truss is placed on a 280mm gap as the problem statement indicates, all three collapse mechanisms influence the strength of the bridge. The main parameters that influence the strength of the bridge are the number of bumps and the contact angle; to simplify the problem, the frictional coefficient, placement of the weights, and humidity are controlled to be constant.

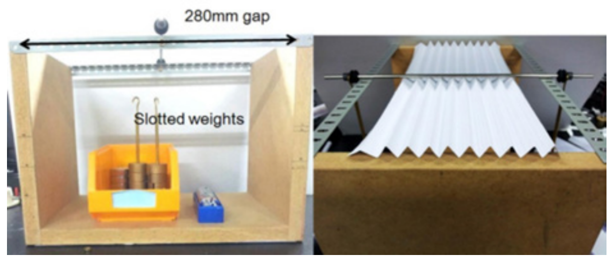

Fig. 15: A paper truss is placed on 280mm gap and the slotted weights are used to control the load on the bridge.

In the Fig. 15, the slotted weights are placed at the two ends of the metal bar that presses down at the middle of the bridge. The wooden plates are used to construct a 280mm gap.

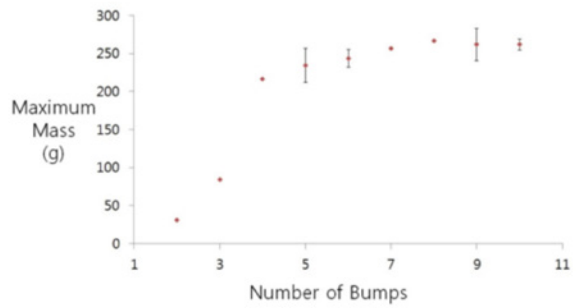

Fig. 16: The strength of the bridge is maximized when the number of bumps is 8.

The number of bumps(n) increases from 2 to 10 whereas the contact angle maintained 50°. The result is shown in the Fig. 16. As n increases from 3 to 4, the strength suddenly increases. This sudden increase of strength is also observed in the Fig. 13, which is explained by the mass of the paper sheet itself. When the number of bumps increases up to 8, the decrease of necking effect is dominant. However, when it exceeds 8, the buckling effect becomes more dominant. Therefore, the maximum strength is achieved when n=8.

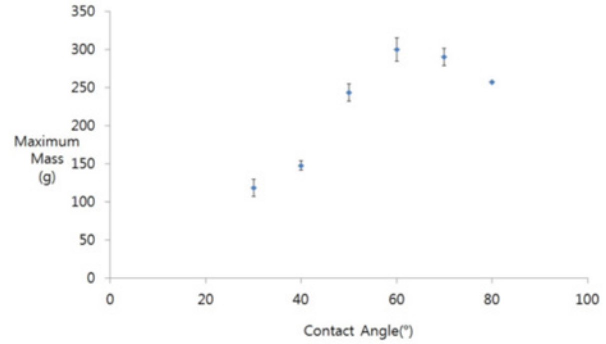

Fig. 17: The strength of the bridge is maximized when the contact angle is 60°.

Next, the contact angle of the truss with 6 bumps is changed from 30° to 80°, by 10°. The result is shown in the Fig. 17. As the contact angle increases up to 60°, the strength increases, since the decrease of sliding effect becomes dominant. As the angle exceeds 60°, the strength starts to decreases, since the increase of the necking effect becomes dominant.

Finally, the maximum strength of a truss ($M_{max.truss}$ =306.9g) is achieved when n=8 and θ=60°, which is bigger than the maximum strength of a rolled-into-a-tube style paper bridge. ($M_{max.tube}$ = 296.2g, m=7)

Further Investigation – changing the length ratio in n=2 case

The bridge members of a truss do not have to have the same size and contact angle. Then what happens to a truss if the length of each bridge member is different? In order to simplify the problem, only n=2 truss is dealt

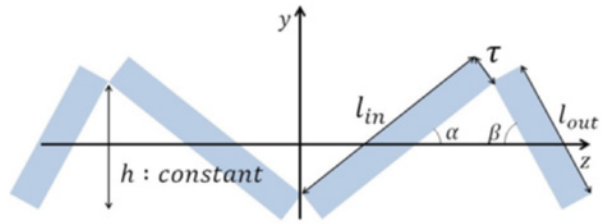

Fig. 18: n=2 truss which has asymmetric bridge

members.

The Fig. 18 shows the n=2 truss with length ratio l_{in} :l_{out}. If the height of the truss is constant, the second moment of area is expressed as the following equation.

$$I_x = \frac{\tau l_{in}^3}{12}(1 + \cos 2\alpha) + \frac{\tau l_{out}^3}{12}(1 + \cos 2\beta) \quad (20)$$

Each of two contact angles is related to the height and the length of the bridge member.

$$l_{in} = h\ sin_{\alpha} \quad (21)$$

$$l_{out} = h\ sin_{\beta} \quad (22)$$

Therefore, the strength of the bridge is relevant to the ratio between α and β. The local minimum of the second moment of area is at the point where $\alpha = \beta$. Also the maximum strength of the bridge can be found from the second moment of area.

$$F_{max} = \frac{4\sigma_{max} I}{Ly} \quad (23)$$

Since the height is constant, the local minimum of the strength is also at the point where $\alpha = \beta$. This result implies that a truss can be stronger if it has asymmetric bridge members.

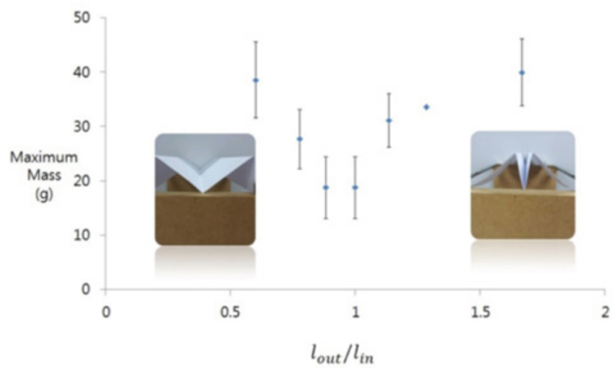

Fig. 19: The strength of the bridge has local minimum when the length ratio is 1.

The Fig. 19 shows how the strength of the bridge changes as the length ratio changes. As the theoretical model predicts, the local maximum is at the point where length ratio is 1.

Conclusion

This research paper considers the number of bumps and the contact angle as the major parameters of strength of an A4 sheet truss. The three types of collapse scenarios – buckling, sliding, and necking - are analyzed to find the maximum weight that a truss structure can hold. The dominance of each mechanism depends on the number of the bumps and the contact angle. As the number of bumps increases up to 8, the decrease of the necking effect strengthens the bridge. As the number of bumps exceeds 8, the increase of buckling effect becomes more dominant and the strength of the bridge decreases. As the contact angle increases up to 60°, the decrease of buckling and sliding becomes and the strength of the bridge increases. After it reaches the local maximum, the strength decreases because the increase of necking effect becomes dominant. The maximum strength of a truss ($M_{max.tube}$ = 306.9g) is achieved when n = 8 and θ = 60°, which is bigger than the maximum strength of a rolled-into-a-tube style paper bridge. ($M_{max.tube}$ = 296.2g, m = 7)

Other than the number of bumps and the contact angle, the ratio between the bridge members also affects the strength of a paper truss. If the height of the bridge is maintained the same, the strength of the bridge, considering the buckling effect, is minimized when the length ratio is 1. The experiment curve verified that there is local minimum point as predicted, yet if the length ratio becomes too big or small, necking and sliding effect cause the decrease of the strength. Though this research only considers the asymmetry of n = 2 truss, it would be able to make a stronger bridge if the asymmetry is applied to the other truss with higher number of bumps.

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