2013 Problem 2 : Elastic Space
Study on gravity in 2-dimensional space through elastic membrane experiment

Abstract

We used a system of a rolling ball on a stretched horizontal membrane as a model system to investigate gravity in 2-dimension (2D). Gauss theorem tells us the gravity force in 2D is inversely proportional to the distance between the two masses, not to the square of the distance. Theoretical analysis of the force on a ball on a stretched membrane indeed shows a force inversely proportional to the distance, with the gravitational constant \( G = \frac{g^2}{2\pi \tau} \) where \( g \) and \( \tau \) are the Earth’s gravitational acceleration and surface tension of the membrane. Experiments on a membrane system also show that the force is inversely proportional to the distance. We also investigated the motion of the ball and found it consistent with simulation results in an inverse field. In particular, we find that the speed of the ball for a circular motion is independent of the orbit radius, and that the apsidal angle for an orbit is about 128°, both of which are explained within the theory.

Keywords

Gravity in two-dimension, Surface tension, Inversely proportional, Symmetry, Apsidal angle.

Introduction

We often use two massive balls placed on a stretched elastic membrane, such as a rubber sheet, in order to demonstrate gravitational effect. The main purpose of such demonstration is primarily to make people easily understand the notion of deformation of the space by mass. However, the characteristic 2-dimensional (2D) motion of such balls placed on a membrane can be quite different from the planetary motion in our 3D world. For example, orbits of balls on a membrane may not be conic sections as in the inverse square field. In order to investigate characteristic differences between motions in gravitational fields in 2D and 3D worlds, we study the motion of a ball on a stretched membrane.

In studying such phenomenon, we first state some principal questions as follows;

- What is the characteristic difference between the spaces in our world and of the elastic membrane?
- What is the expression for the gravitational force on a ball of the rubber membrane system? Is it possible to define the gravitational constant \( G \)?
- What kind of characteristic differences does the motion of the ball in the system exhibit?

In order to answer these three main questions, we first constructed a circular elastic membrane, and then analyzed the system theoretically. Finally, we conducted measurements on the ball motion to investigate different phenomena.

Construction of the Membrane System

In order to effectively analyze such a system, a membrane system has to be constructed first. The system has to meet several conditions for valid experiments. They are

- In order to mimic the real-life centripetal gravitational force with a rotational symmetry, the shape of the membrane must show the same property (physical quantities must depend only on the distance from the center).
- The initial surface tension of the membrane must be large enough so that gravity sag of the membrane is negligible. Also seen in the picture is a small acrylic disc at the center.
- What is the characteristic difference between the gravitation force with a rotational symmetry, the shape of the membrane must show the same property (physical quantities must depend only on the distance from the center).
- In order to meet these conditions, we first decided to use a rubber sheet membrane attached to a circular plastic container as our main system. The rubber sheet, which has high elasticity, is initially uniformly stretched before being glued for high enough surface tension. We also attached a small disc at the center so that we can hang a massive weight at the center of the membrane. This lets the mass of the material to be concentrated at the center with the contact area remaining the same. The constructed system is shown in Fig. 1.

Theoretical Approach

1. Classical definition of gravitational force

The most fundamental approach is to know the equation of motion of the test mass. The gravitational force in general can be expressed as

\[
F = G \frac{M \cdot m \cdot r}{r^2}
\]

where \( G \), \( M \), \( m \), and \( r \) are gravitational constant, two masses and the distance between them, respectively. The expression implies that if we are able to define the gravitational force as a function of distance between two masses \( M \) and \( m \), if the proportionality power of two masses \( n \) is the same, we can define the gravitational constant \( G \). The reason why the power \( n \) must be the same is because of the symmetry in the Newton’s third law. Both \( M \) and \( m \) can act as the central mass, which lets both \( M \) and \( m \) play the same role. In addition, the system must have a center-symmetric shape (gravitational force depending only on the distance from the center) in order to define a gravitational constant. This is why we need to use a circular membrane.
2. Characteristics of the system

For further systematic analysis, we must clarify the characteristics of our system and define relevant notations. Some characteristics of the system are as follows (see Fig. 2):

- Circular elastic membrane is deformed by the central mass \( M \) while the test mass \( m \) is light enough not to deform the membrane
- The test mass shows approximately a 2D motion in a shallow well
- The 'gravity' is induced by the Earth's gravity & the slope of the membrane

Due to these characteristics, it is now obvious that the gravitational force within the system has linear relationship with the gravitational force and the distance between \( m \) and \( M \) in 2D, not to deform the membrane.

3. Gauss' flux theorem for gravity

Gauss' flux theorem for gravity is mainly used for 3D case, which is \([1,2]\)

\[ \oint \mathbf{g} \cdot dA = -4\pi GM \]  
(2)

In 2D, however, the form of Gauss' flux theorem becomes

\[ \oint \mathbf{g}_{2D} \cdot dl = -2\pi GM \]  
(3)

Simplifying equation (3) by exploiting the rotational symmetry, we get

\[ g_{2D} = -\frac{GM}{r} \]  
(4)

or

\[ F_{2D} = -\frac{GMm}{r} \]  
(5)

Through the application of Gauss' flux theorem for gravity, we find that the magnitude of the gravitational force is inversely proportional to the distance between \( M \) and \( m \) in 2D, not to the distance square as in 3D. The only parameter we do not know is the gravitational constant \( G \).

Since the actual motion of the small mass \( m \) is induced by the Earth's gravitational force and the slope of the membrane, we will be able to define the gravitational constant \( G \) once we can calculate the membrane profile \( h \) (displacement from the initial position).

4. Definition of the gravitational constant \( G \)

There are mainly three ways to predict the membrane profile for extracting the gravitational constant \( G \). The three methods are

- Force analysis method
- Rubber sheet model
- Comparison of the rubber sheet model and the gravitational Poisson equation

The first and second methods are quite similar. However, the difference between these two methods is that the first method leads to the result through a macroscopic view while the second method through a microscopic one. Each process of defining the gravitational \( G \) will be introduced in the next section.

Before we move on to the force analysis, one needs to understand the concept of 'surface tension'. The fundamental definition of surface tension is 'force per unit length' which can be measured, for rectangular rubber strips, by dividing the applied force by the width of the point of action. How we measure the surface tension of the membrane will be described later in the measurement section.

a) Force analysis method: One way to predict the membrane profile is by conducting a (vertical) force analysis when the central mass and the rubber membrane remain in a static equilibrium. Assuming that the surface tension is constant throughout the membrane, the force equilibrium equation can be expressed as (see Fig. 3)

\[ \tau \frac{\partial h}{\partial r} = -\sigma \]  
(6)

where \( \tau \) is a constant. Using

\[ \tau \frac{\partial h}{\partial r} = -\sigma \]  
(6)

the solution of the Laplace operator \( \nabla^2 \) in cylindrical coordinates is given by

\[ \nabla^2 h = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial h}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 h}{\partial \theta^2} + \frac{\partial^2 h}{\partial z^2} = 0 \]  
(11)

Due to the symmetry in the system, we can neglect the term concerning \( \theta \). Solution to equation (11) without the \( \theta \) term is

\[ h = \frac{\ln r}{R} \]  
(9)

where \( R \) stands for the radius of the rubber sheet membrane and is used to determine the integration constant. The assumption we used for equation (9) is valid as far as the vertical displacement \( h \) is small and the surface tension remains constant.

b) Rubber sheet model: The rubber sheet model is given by[3]

\[ \tau \frac{\partial h}{\partial r} = -\sigma \]  
(10)

where \( h \) is the vertical displacement of the membrane. Since the mass of the rubber sheet is small enough not to affect the sheet profile, we can neglect the mass density \( \sigma \) in the equation. Therefore, the right hand side of the equation now equals to zero. The solution of the Laplace operator \( \nabla^2 \) in cylindrical coordinates is given by

\[ \nabla^2 h = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial h}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 h}{\partial \theta^2} + \frac{\partial^2 h}{\partial z^2} = 0 \]  
(11)

Fig. 4. Force analysis at the boundary of the central disk.

\[ G = \frac{g_e^2}{2\pi \tau} \]  
(8)

We can also obtain the membrane profile from equation (7). Assuming \( \tau \) to be small enough, \( \sin \theta \approx \tan \theta \), which enables us to predict the membrane profile. The result is as follows,

\[ h = \frac{\tan \theta}{r} \]  
(9)

Comparing (5) and (7) gives us the expression for the gravitational constant,
\( \tau = 2\pi \sigma \frac{\partial h}{\partial r} = Mg \) \hfill (12)
\( c = \frac{M g e}{2 \pi} \) \hfill (13)

The gravitational force
\( F = mg \sin \theta = m g \frac{d^2 h}{dr^2} = \frac{GMm}{r^2} \) gives the same gravitational constant \( G \) as in (8).

\( r \) is the sound frequency. To determine \( r \) and \( L/L = 0 \) (21)
\( \Phi = -\frac{2\pi}{\alpha} \) \hfill (15)

Since the product of \( g \) and \( h \) in (14) is equivalent to the gravitational potential and \( \Phi \) in (15) is the gravitational potential, comparison of (14) and (15) gives us
\( \frac{\sigma g e^2}{\tau} = 2\pi \sigma G \) or \( \frac{G}{2\pi} = \frac{\sigma g e^2}{\tau} \) \hfill (16)

which is again the same as that in (8). The gravitational force being \( F_{DD} = \frac{\sigma g e^2 M}{2\pi} \), we can obtain the gravitational potential by integrating the gravitational force and further obtain the membrane profile \( h \). Results obtained by all three different methods come out to be the same, as they should be.

**Experimental Measurements on Static Properties**

Now that we have thoroughly analyzed our system with various methods, we must now verify whether our assumptions and analysis are valid or not. We will first measure the surface tension of the membrane and the membrane profile. Then, we will measure the gravitational force induced by the central mass and finally make predictions on the motion of the test mass through simulations.

**1. Surface tension measurements**

Looking at equations (7) and (8), it is obvious that being able to precisely measure the surface tension \( \sigma \) is an extremely important factor since the surface tension is the only parameter that we do not know. Therefore, we conducted experimental measurements of the surface tension in two different ways.

a) Conventional method: The classical method of measuring the surface tension is to use the definition of surface tension. That is,
\( \tau = \frac{F}{\text{width}} = k \frac{\Delta L}{L} \) \hfill (17)

where \( k \) stands for the elasticity of the rubber membrane which has a dimension N/m, and \( L \) and \( \Delta L \) are the initial and elongated lengths, respectively.[3]

We first measured the elasticity of the rubber sheet by utilizing equation (17). We used a rubber strip which has a length of 9 cm, and a width of 15 cm (see Fig. 5). The magnitude of the force we have applied on the rubber strip was about 41.6 N, and the elongated length was about 4 mm. Plugging all these into equation (17) gives us the elasticity of the rubber \( k \) of about 6240 N/m.

b) Surface tension from sound frequencies:

Another method to measure the surface tension is by exploiting the characteristic sound frequency of the rubber membrane. The relationship between the surface tension and the membrane frequency can be derived in a similar way to that between the tension of a rope and its frequency.[4]

The equation of motion in this situation is,
\( \tau \Phi h = \sigma \frac{\partial^2 h}{\partial t^2} = 0 \) \hfill (18)

where \( \Phi \) is the surface mass density. Substituting \( h(r, \theta, t) = S(r) \Theta(\theta) T(t) \) we get,
\( \frac{1}{k} \Psi^2 S = \frac{1}{2} \frac{\partial^2 S}{\partial r^2} + \frac{T}{k} = -k^2 \) \hfill (19)

Solving this equation for \( T(t) \) we get,
\( T(R) \sim \sin \) \hfill (20)

where \( \sim \) is the sound frequency. To determine \( k \), we solve for \( S(r, \theta) \)
\( \Psi^2 S + k^2 S = 0 \) \hfill (21)

We use separation of variables again to express \( S(r, \theta) \) as \( L(r)\Theta(\theta) \). The solution of \( L(r) \) in this equation is known to have the form of a Bessel functions. Due to the boundary conditions, the value \( L(R) \) must equal to 0. For \( L(R) \) to be zero, the product of \( k \) and \( R \) must be one of the Bessel zeroes \( (\alpha_{mn}) \).[5] Now we can state
\( \frac{\alpha_{mn}}{R} \sim \sqrt{\frac{2\pi f}{\alpha}} \) \hfill (22)

Substituting the first Bessel zero into equation (22), we obtain
\( \tau = \sigma \cdot \left( \frac{2\pi R f}{2.404} \right)^2 \) \hfill (23)

The fundamental frequency was measured by simply tapping the membrane and analyzing the sound profile through the Fourier analysis (see Fig. 6). The measured fundamental frequency was about 11 Hz. In combination with the values of \( \sigma = 2.75 \text{ kg/m}^2 \) and \( R = 29 \text{ cm} \), equation (23) gives us \( \tau = 191 \text{ N} \). Comparing this with the value measured through the conventional method, we see that the two measurements show a fairly good agreement. Due to the larger error bar for the classical method, we decided to use the surface tension value measured through the sound analysis.

**2. Membrane profile measurement**

With the exact value of the surface tension, the membrane profile can be calculated theoretically. We then need to compare the calculated and measured membrane profiles. To experimentally measure the membrane profile, we used a vernier caliper in order to measure the vertical displacement of the membrane as a function of \( r \). We also varied the central mass \( M \) for each experiment. The experimental setup and the results are shown in Fig. 7. In the results, we can see...
We will see demonstrations of gravity, we often see a small ball ‘rolling’ on the membrane. Now that we know it is a motion induced by a 2D gravity, we may attempt to predict its orbit. Due to the complexity of the equation of orbit, we decided to conduct computational simulations using a program called Interactive Physics. By setting the force field as below, we could effectively predict the 2D motion within the system.

\[
f = \frac{5 \, G \, m \, M}{r^2} - \mu \, m \, g \, \frac{\dot{r}}{v}
\]

(24)

The second term was added to account for the frictional force and damping effect from the membrane deformation. The friction constant \( \mu \) was obtained experimentally by rolling a ball on the tilted rubber membrane. In order to reduce the damping effect, we used light-weight balls. Since the computational simulation considers only a pure translational motion while the actual motion is a rolling motion, a factor \( \frac{5}{7} \) is multiplied in order to compensate the rolling effect. [2] However, for the case of spiral motion in particular which we will discuss later on, \( \frac{5}{7} \) was removed from equation (24).

We also used a program called Kinovea in order to track the path of the rolling ball, and measure its velocity and position as a function of time. The simulation and experimental results are shown in Fig. 10. Even though there are some detailed differences, the overall shape is quite similar between the two results.

The relationship between the mechanical energy and its orbit has long been an interest in the field of mechanics. Studies have been done for different types of central forces with a form of \( -\frac{c}{r^n} \). We will investigate the case of \( n = 1 \).

Since the small mass shows a rotational motion revolving around the central mass, not only does the small mass have potential energy and radial kinetic energy, but it also has a rotational kinetic energy. Adding up the potential and rotational kinetic energies, we get an ‘effective potential’ expressed as [6]

\[
V(r) + \frac{L^2}{2m_1 r^2} = \frac{m_2}{r}
\]

(25)

Adding up the radial kinetic energy and the effective potential energy, we can express the total mechanical energy as

\[
E_{\text{tot}} = \frac{1}{2}mv^2 + \frac{L^2}{2m_1 r^2} + V(r)
\]

(26)

Calculating \( V(r) \) for this system we now obtain,

\[
E_{\text{tot}} = \frac{1}{2}mv^2 + \frac{L^2}{2m_1 r^2} + \frac{m_2}{r}
\]

(28)

The relationship between the total energy and the trajectory of the revolving mass in 3D world is known to be the following.

- \( E_{\text{tot}} \geq 0 \): Hyperbolic orbit
- \( E_{\text{tot}} = 0 \): Parabolic orbit
- \( E_{\text{tot}} = E_{\text{min}} \): Elliptical orbit
- \( E_{\text{tot}} < E_{\text{min}} \): Circular orbit

For 2D, there is no unbound motion (that is, \( m \) cannot escape) because the potential energy increases forever as \( r \) increase. Therefore, there are only two kinds of orbits in 2D.

- \( E_{\text{tot}} = E_{\text{min}} \): ‘Elliptical’ orbit
- \( E_{\text{tot}} < E_{\text{min}} \): Circular orbit

Even though the exact shape of the ‘elliptical’ orbit in 2D is different from that in 3D, we still use the
word ‘elliptical’ because the orbit is analogous to the elliptical orbit induced by 3D gravity. The relationship between the trajectory and the energy was investigated experimentally, and the results are shown in Fig. 11.

3. r independent speed for circular motions

Another characteristic phenomenon of this system can be found through the force analysis. The gravitational force acting as the centripetal force in this system and the force equation is expressed as

\[ \frac{mv^2}{r} = \frac{GMm}{r} \]  

(28)

Here, the inverse of \( r \) cancels each other, and we obtain

\[ v = \sqrt{GM} \]  

(29)

This is quite an interesting result because \( v \) is totally independent from the radial distance. Therefore, we can conclude that wherever we start rolling the ball, a circular motion is obtained if we set the speed at such value. However, due to the existence of damping effect, the motion will rather show a spiral orbit. For reproducible experiments, we used an electromagnet and a steel ball placed on a slope. The results are shown in Fig. 12. Plotting the speed as a function of time, we can see that the speed is constant throughout the motion. Due to the damping effect, the radial distance changes as time passes and therefore the graph also indicates that the speed is constant regardless of the radial distance. Plugging in our theoretical prediction into the computational simulation program, we can also see a spiral motion.

3. Apsidal angle

If we compare Fig. 10 with an elliptical orbit in 3D, the orbits are quite different. It is obvious that the difference in shape should be caused due to the difference in the force type. An effective way to analyze its orbit is to investigate the ‘apsidal angle’ of an orbit.[1] An apsidal angle is the angle between the two lines connecting the focal point and the two consecutive apsides. The apsis is either the minimum distance or the maximum distance from the focal point within the orbit. The apsidal angle can be calculated through the ratio of the frequency of the radial oscillation and the unperturbed circular motion. The apsidal angle for central force with a general form of \( \frac{1}{r} \) is given as \( \frac{\pi}{2n} \). This actually indicates that only central forces of \( n=2 \) or \( -1 \) has a closed orbit. Orbits in Fig. 10 for which \( n=1 \) actually are not closed orbits. Substituting \( n=1 \) into the equation, we find the apsidal angle to be \( 127.28^\circ \). Plotting the orbit on a polar coordinate, we found the actual apsidal angle to be \( 127.9^\circ \). The discrepancy is believed to be due to the friction. The experimental results are shown in Fig. 13.

![Fig. 12. Experimental setup for circular orbit investigation and r independent speed.](image)

![Fig. 13. Apsidal angles for orbits in 2D. Simulation (left) and experimental results (right).](image)

4. Radial oscillation

If we plot the radial distance of the test mass as a function of time, we can see that the ball shows an oscillatory motion. We can predict the period of the radial oscillation by calculating the period between the consecutive apsides. For a central force of \( f(r) \), the period is given as

\[ T_r = 2\pi \sqrt{\frac{m}{\frac{GM}{r}}} \]  

(30)

Substituting \( \frac{GM}{r} \) for \( f(r) \), we obtain

\[ T_r = 2\pi \sqrt{\frac{m}{GM}} \]  

(31)

Since the radial distance varies throughout the motion, we must average the radial distance of each oscillation. The process of averaging the radial distance is quite simple due to the fact that the period shows a linear response to the radial distance. The experimental results are in Fig. 14. Looking at the right side of the graph, we find some discrepancy due to the increase of the slope in which case \( \cos \theta \approx 1 \) approximation is no longer valid.

![Fig. 14. Radial distance vs time graph (left) and comparison between the measured and theoretical values for the period (right).](image)

Summary & Discussion

In order to investigate characteristic differences between balls rolling on a stretched and deformed elastic membrane and planetary motions in 2D, we first constructed a system where the gravitational force can be well defined. We then defined the form of the gravitational force with consideration of the characteristics of the membrane by using the Gauss’ flux theorem as well as a force analysis. In order to prove the validity of the theory, we conducted measurements such as surface tension, membrane profile, and gravitational force. In addition, we performed computational simulation of 2D motion. Once we had a concrete proof of the theory, we proceeded to investigate characteristic phenomena in the system of 2D gravity. Throughout the investigation, it became clear that a rolling ball on a stretched membrane shows characteristic motions predicted under 2D gravity. The measurements showed an excellent agreement with the theory, and different phenomena within the system have been investigated. Through the theoretical and experimental studies, we were able to explain the phenomena both qualitatively and quantitatively, which leads us to have a deeper understanding of the 2D gravity.

For future experiments, we may try other types of stretched horizontal membranes and see if other characteristic phenomena can exist with different types of the central force. We may also conduct experiments on the same type of membrane with a larger vertical displacement and investigate what differs from our original experiment. Furthermore, we can conduct experiments on rigid surfaces with a similar shape to the deformed membrane to investigate the damping effect.
Abstract
This is the first problem of 2012 IYPT, it’s a dynamics problem. This article analysis the problem both in theory and experiments.

Keywords
paper bridge 280mm strength

Introduction
The problem is: It is more difficult to bend a paper sheet when it is folded in “accordion style” or rolled into a tube. Using a single A4 sheet and a small amount of glue, if required, construct a bridge spanning a gap of 280 mm. Introduce parameters to describe the strength of your bridge, and optimise some or all of them. The key words of the problem include: A4 paper, a small amount of glue, and 280mm gap. The problem asks us to build a certain kind of paper bridge and use parameters to describe the strength of the bridge and optimize it.

Theory
First look at the potential parameters. The size of the paper is fixed to A4, but the type of the paper can vary. Of course, the stronger the paper is, the stronger the bridge is (Fig. 1 and Fig. 2). For the glue, we need to choose the glue which can stick well and is not fragile.

References